will be true of Series IV, obtained by striking out the terms containing at least a 9's and b 8's and c 7's; this is the case $d = e = \cdots = j = 0$.

It will be sufficient, as before, to show that the terms not occurring in series III that have been introduced into series IV form, of themselves, a convergent series. To this end it is merely necessary to notice that none of these terms contains more than $(c - 1)$ 7's; for since, by our special case, the series consisting of *all* terms that do not contain more than $c - 1$ 7's is convergent, *à fortiori* will any series made up of a selection of such terms be convergent, since all these terms are positive.

This completes the proof except for the special treatment demanded by the digit 0. It will, perhaps, suffice to indicate how Dr. Kempner's proposition would be extended to cover this case, leaving to the reader the proof of our "special case" for the digit 0. We are to show, then, that the series obtained from the harmonic by striking out terms containing the digit 0 converges. There are 9^n numbers of n digits not containing 0; the sum of their reciprocals is less than $9^n/10^{n-1}$; and our series is less than $\Sigma 9^n/10^{n-1}$, *i. e.*, 90.

2. Let us return to Dr. Kempner's series, namely that obtained from the harmonic series by striking out terms whose denominators contain the digit 9. This series converges; what is its sum? Dr. Kempner merely shows that it is less than 90. Its value actually lies between 22.4 and 23.3. This result may be obtained with no great amount of labor, and a considerably closer approximation might be reached, if desired, by the methods here employed.

Consider those terms of our series that have two digits in the denominator:

$$a_2 = \frac{1}{10} + \frac{1}{11} + \cdots + \frac{1}{18} + \frac{1}{20} + \frac{1}{21} + \cdots + \frac{1}{28} + \cdots + \frac{1}{80} + \frac{1}{81} + \cdots + \frac{1}{88}.$$

Compare with these the terms with three digits in the denominator. The first nine of them, $1/100 + 1/101 + \cdots + 1/108$, are each less than or equal to $1/100$, and their sum is less than $9/100$, that is, less than $9/10$ of $1/10$, the first number of a_2. In the same way the next set of nine, $1/110 + 1/111 + \cdots + 1/118$, is less than $9/10$ of $1/11$, the second number of a_2 and so on: the sum of all the terms with three digits is less than $9/10$ of a_2. Similarly, the sum of all terms with four digits is less than $(9/10)^2$ of a_2; and in general, the sum of all terms with n digits is less than $(9/10)^{n-2}$ of a_2. Therefore the sum of our series is less than

$$1 + \frac{1}{2} + \cdots + \frac{1}{8} + \left[1 + \frac{9}{10} + \left(\frac{9}{10}\right)^2 + \cdots\right] a_2 = 1 + \frac{1}{2} + \cdots + \frac{1}{8} + 10a_2.$$

The value of a_2 may be quickly computed with the help of a table of reciprocals. We have:

$$1 + \tfrac{1}{2} + \cdots + \tfrac{1}{8} < 2.72$$
$$a_2 < 2.058; \quad 10a_2 < 20.58$$

The sum of the series is less than 23.3

To obtain an inferior limit for the sum of the series, we show in the same way

that the sum of the terms with n digits in the denominator is greater than $(9/10)^{n-2} \cdot a_2'$, where

$$a_2' = \frac{1}{11} + \frac{1}{12} + \cdots + \frac{1}{19} + \frac{1}{21} + \frac{1}{22} + \cdots + \frac{1}{29} + \cdots + \frac{1}{81} + \frac{1}{82} + \cdots + \frac{1}{89};$$

and, therefore, the series is greater than

$$1 + \tfrac{1}{2} + \cdots + \tfrac{1}{8} + a_2 + [9/10 + (9/10)^2 + \cdots]a_2',$$

that is, greater than

$$1 + \tfrac{1}{2} + \cdots + \tfrac{1}{8} + a_2 + 9a_2',$$

which turns out to be greater than 22.4.

A still closer approximation may be found by starting with a_3 and a_3', that is, with the terms having three digits in the denominator and with

$$a_2' = \frac{1}{101} + \cdots + \frac{1}{109} + \frac{1}{111} + \cdots + \frac{1}{119} + \cdots + \frac{1}{881} + \cdots + \frac{1}{889}.$$

ON THE MATRIX EQUATION $BX = C$.[1]

By H. T. BURGESS, University of Wisconsin.

Section 1. To Find the Matrix X. The problem is to calculate the elements of the matrix X to satisfy the matrix equation $BX = C$:

$$\begin{Vmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{Vmatrix} \begin{Vmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{Vmatrix} = \begin{Vmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{Vmatrix}.$$

If we compute the matrix product BX, we get

$$BX = \begin{Vmatrix} \Sigma b_{1\epsilon}x_{\epsilon 1} & \Sigma b_{1\epsilon}x_{\epsilon 2} & \cdots & \Sigma b_{1\epsilon}x_{\epsilon n} \\ \Sigma b_{2\epsilon}x_{\epsilon 1} & \Sigma b_{2\epsilon}x_{\epsilon 2} & \cdots & \Sigma b_{2\epsilon}x_{\epsilon n} \\ \cdot & \cdot & \cdots & \cdot \\ \Sigma b_{n\epsilon}x_{\epsilon 1} & \Sigma b_{n\epsilon}x_{\epsilon 2} & \cdots & \Sigma b_{n\epsilon}x_{\epsilon n} \end{Vmatrix},$$

where the summation runs for $\epsilon = 1, 2, \cdots, n$.

The conditions to be fulfilled are obtained by setting the elements of the product BX equal to the corresponding elements of C. Taking these by columns we get the following n-sets of simultaneous linear equations:

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[1] For the elementary properties of matrices the reader may conveniently consult Bôcher's *Introduction to Higher Algebra*, using the index to find the appropriate sections.

$$\text{I.}\qquad \begin{aligned} b_{11}x_{1\kappa} + b_{12}x_{2\kappa} + \cdots b_{1n}x_{n\kappa} &= c_{1\kappa},\\ b_{21}x_{1\kappa} + b_{22}x_{2\kappa} + \cdots b_{2n}x_{n\kappa} &= c_{2\kappa},\\ \cdots\cdots\cdots\

we write

$$\left\|\begin{matrix} 1 & 5 & 1 & 18 & 11 & 3 \\ 3 & 4 & 2 & 20 & 11 & 7 \\ 2 & 1 & 3 & 10 & 4 & 8 \end{matrix}\right\|.$$

Reducing the first column by (3), we get

$$\left\|\begin{matrix} 1 & 5 & 1 & 18 & 11 & 3 \\ 0 & -11 & -1 & -34 & -22 & -2 \\ 0 & -9 & 1 & -26 & -18 & 2 \end{matrix}\right\|.$$

Reducing the third column by (3), we get

$$\left\|\begin{matrix} 1 & 14 & 0 & 44 & 29 & 1 \\ 0 & -20 & 0 & -60 & -40 & 0 \\ 0 & -9 & 1 & -26 & -18 & 2 \end{matrix}\right\|.$$

Dividing the second row by -20 by use of (2), and reducing the second column by (3), we get

$$\left\|\begin{matrix} 1 & 0 & 0 & 2 & 1 & 1 \\ 0 & 1 & 0 & 3 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 & 2 \end{matrix}\right\|. \quad \text{Hence } X = \left\|\begin{matrix} 2 & 1 & 1 \\ 3 & 2 & 0 \\ 1 & 0 & 2 \end{matrix}\right\|.$$

The method applies to $XB = C$ by use of the conjugates, for $B'X' = C'$.

Section 2. To Find the Inverse of a Matrix. A very useful application occurs when C is replaced by the unit matrix I, for in this case X becomes the inverse of B. The amount of work required to calculate the inverse of a matrix by this method is practically the same as that required to compute one of its elements by the ordinary method.

Illustration: To compute the inverse of the matrix

$$B = \left\|\begin{matrix} 1 & -1 & -5 \\ -1 & 3 & -10 \\ -1 & 0 & 12 \end{matrix}\right\| \text{ we write } \left\|\begin{matrix} 1 & -1 & -5 & 1 & 0 & 0 \\ -1 & 3 & -10 & 0 & 1 & 0 \\ -1 & 0 & 12 & 0 & 0 & 1 \end{matrix}\right\|.$$

Reducing column one by (3):

$$\left\|\begin{matrix} 1 & -1 & -5 & 1 & 0 & 0 \\ 0 & 2 & -15 & 1 & 1 & 0 \\ 0 & -1 & 7 & 1 & 0 & 1 \end{matrix}\right\|.$$

Interchanging rows two and three by (1), changing signs in row two by (2), and reducing column two by (3):

$$\begin{Vmatrix} 1 & 0 & -12 & 0 & 0 & -1 \\ 0 & 1 & -7 & -1 & 0 & -1 \\ 0 & 0 & -1 & 3 & 1 & 2 \end{Vmatrix}.$$

Changing the signs in the last row by (2) and reducing column three by (3):

$$\begin{Vmatrix} 1 & 0 & 0 & -36 & -12 & -25 \\ 0 & 1 & 0 & -22 & -7 & -15 \\ 0 & 0 & 1 & -3 & -1 & -2 \end{Vmatrix}.$$

Hence,

$$B^{-1} = \begin{Vmatrix} -36 & -12 & -25 \\ -22 & -7 & -15 \\ -3 & -1 & -2 \end{Vmatrix}.$$

CENTERS OF SIMILITUDE AND THEIR N-DIMENSIONAL ANALOGIES.

By BANCROFT HUNTINGTON BROWN, Brown University.

1. In the January, 1915, issue of the MONTHLY[1] "Centers of similitude of circles and certain theorems attributed to Monge" were discussed. The theorems there given are the following:

(*A*) *The six centers of similitude of three coplanar circles lie by threes on four straight lines.*[2]

(*B*) *The vertices of the six common tangent cones of three spheres, taken in pairs, lie by threes on four straight lines.*

(*C*) *Given any four spheres in space fixed in magnitude and position, and the six cones tangent to them in pairs, externally; then the six vertices lie in a plane and indeed on four straight lines in the plane. If the six other tangent cones be drawn, then their vertices lie by threes in planes*[3] *with threes of the first group.*

It was shown:

(1) That theorem (*A*) was, in all probability, known to the Greeks of two thousand years ago;

(2) That Fuss found, with regard to the *external* centers of similitude of coplanar circles, that:

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[1] R. C. Archibald, THE AMERICAN MATHEMATICAL MONTHLY, Vol. XXII, pp. 6–12.

[2] Symbolically, if $E_{m,n}$ denote the external, and $I_{m,n}$ the internal centers of similitude of the circles C_m and C_n $(m, n = 1, 2, 3.\ \ m \gtrless n)$, the following groups of points are collinear:

$$E_{1,2},\ E_{1,3},\ E_{2,3};\ E_{1,2},\ I_{1,3},\ I_{2,3};\ I_{1,2},\ E_{1,3},\ I_{2,3};\ I_{1,2},\ I_{1,3},\ E_{2,3}.$$

[3] These planes have been called "planes of similitude."

(*D*) *For n coplanar circles, the $n(n-1)/2!$ external centers of similitude are situated, in general, by threes on $n(n-1)(n-2)/3!$ different straight lines;*[1]

(3) That the centers of similitude of the spheres of theorem (*C*) lie by sixes on eight different planes. Further generalization of these results does not seem to have been published. It is the object of this paper to give indications of such generalizations.

2. Consider first the number of centers of similitude of n coplanar circles; any circle combined with each of the $(n-1)$ remaining circles determines 2 centers of similitude, and since in this way each pair of centers of similitude is enumerated twice we arrive at the result:

n coplanar circles have, in general, $n(n-1)$ centers of similitude.[2]

Three circles may be selected from n circles in $n(n-1)(n-2)/3!$ different ways. With any such set of three circles we may by theorem (*A*) associate a group of four lines on which the centers of similitude lie by threes. Hence we have:

(*E*) *The $n(n-1)$ centers of similitude of n coplanar circles are situated, in general, by threes on $4n(n-1)(n-2)/3!$ different straight lines.*

Some particular cases may be noted: (*a*) If three (or more) circles have two common external tangents, then three (or more) of the centers of similitude coincide and *any* line through this point would contain at least three centers of similitude; moreover all of the centers of similitude lie on a single line. (*b*) Again, if two or more pairs of circles are in perspective[3] from the same point, then at least four axes of similitude coincide. For if $E_{1,2}$ and $E_{3,4}$ are coincident, this point is collinear with $E_{1,4}$ and $E_{2,4}$, with $E_{2,4}$ and $E_{2,3}$, and with $E_{1,3}$ and $E_{1,4}$; hence, these three lines coincide with the fourth line containing $E_{2,3}$, $E_{2,4}$, and $E_{3,4}$. Similar reasoning may be employed if $E_{1,2}$ and $I_{3,4}$ coincide. (*c*) If, of two pairs of equal circles, the lines of centers are parallel, the circles may be said to be in perspective from a point at infinity, in the direction of their lines of centers, and their external centers of similitude coincide at infinity. This is then a particular case of the preceding. (*d*) If three or more circles are tangent at one point, since this point of tangency is a center of similitude (external or internal) of any two of the circles, *any* line through this point contains at least three (coincident) centers of similitude; and, as in the case of every set of coaxial circles, all the centers of similitude are collinear. (*e*) If, of three circles, at least two are concentric, the six (or five) centers of similitude are collinear.

3. Monge was the first to give theorem (*C*) and he indicated five such planes:

$$E_{1,2},\ E_{1,3},\ E_{1,4},\ E_{2,3},\ E_{2,4},\ E_{3,4};\quad E_{1,2},\ E_{1,3},\ I_{1,4},\ E_{2,3},\ I_{2,4},\ I_{3,4};$$

$$E_{1,2},\ I_{1,3},\ E_{1,4},\ I_{2,3},\ E_{2,4},\ I_{3,4};\quad I_{1,2},\ E_{1,3},\ E_{1,4},\ I_{2,3},\ I_{2,4},\ E_{3,4};$$

$$I_{1,2},\ I_{1,3},\ I_{1,4},\ E_{2,3},\ E_{2,4},\ E_{3,4}.$$

[1] These straight lines have been called "axes of similitude."

[2] If the circles (spheres) are equal and non-concentric, one center of similitude is at infinity, and the other bisects the line joining their centers. If two circles (spheres) are concentric we may say that the two centers of similitude coincide with the common center of the circles (spheres) or that one center of similitude coincides with the common center, the other being indeterminate.

[3] Any two circles are in perspective from either center of similitude.

There are three others, which he overlooked, making eight in all:

$I_{1,2}, I_{1,3}, E_{1,4}, E_{2,3}, I_{2,4}, I_{3,4}$; $I_{1,2}, E_{1,3}, I_{1,4}, I_{2,3}, E_{2,4}, I_{3,4}$; $E_{1,2}, I_{1,3}, I_{1,4}, I_{2,3}, I_{2,4}, E_{3,4}$.

That, in general, not more than six such centers are coplanar, and that not more than eight such planes exist, may be shown as follows: The six points in a plane are L's (I's or E's) with the six *different* pairs of suffixes. $E_{1,2}$ and $I_{1,2}$ cannot lie in one of these planes, for, if they did, the centers of spheres 1 and 2, and then of all the spheres would lie in this plane. Hence, we can determine all the planes by taking three L's with the same suffixes, *e. g.*, 1, 2, 1, 3, 1, 4; that is, in $2 \times 2 \times 2$, or eight, and only eight, different ways.

If now we consider n spheres, having $n(n-1)$ centers of similitude, we can choose 4 spheres in $n(n-1)(n-2)(n-3)/4!$ ways. And since with each of these sets of four spheres we can associate 8 planes containing 6 centers of similitude, we have

(*F*) *The* $n(n-1)$ *centers of similitude of* n *spheres lie, in general, by sixes, in* $8n(n-1)(n-2)(n-3)/4!$ *planes.*

4. Consider now five four-dimensional hyper-spheres. Four points $L_{i,j}$ determine a hyper-plane in which six other centers of similitude lie. For example, if we take $E_{1,2}$, $I_{1,3}$, $E_{1,4}$, $E_{1,5}$; applying theorem (*A*) to spheres and hyper-spheres, by considering the circles of plane sections, we know $E_{1,2}$, $I_{1,3}$, $I_{2,3}$ are collinear; also $E_{1,2}$, $E_{1,4}$, $E_{2,4}$; $E_{1,2}$, $E_{1,5}$, $E_{2,5}$; $I_{1,3}$, $E_{1,4}$, $I_{3,4}$; $I_{1,3}$, $E_{1,5}$, $I_{3,5}$; and $E_{1,4}$, $E_{1,5}$, $E_{4,5}$. Hence $E_{1,2}$, $I_{1,3}$, $E_{1,4}$, $E_{1,5}$, $I_{2,3}$, $E_{2,4}$, $E_{2,5}$, $I_{3,4}$, $I_{3,5}$, $E_{4,5}$ lie in one hyper-plane. By a proof similar to the one used in the case of spheres, (1) not more than 10 centers of similitude can, in general, lie in a hyper-plane, and (2) there can, in general, be 16, and only 16, such hyper-planes, since L_{ij} can be chosen in only $2 \times 2 \times 2 \times 2$ or 16 ways. We have then

(*G*) *The* 20 *external and internal centers of similitude of* 5 *four-dimensional hyper-spheres lie, in general, by tens in* 16 *hyper-planes.*

From among n hyper-spheres we may choose 5 in $n(n-1)(n-2)(n-3)(n-4)/5!$ different ways. And since each of these sets of hyper-spheres has the property of theorem (*G*), in general, there follows

(*H*) *The* $n(n-1)$ *centers of similitude of* n *four-dimensional hyper-spheres lie, in general, by tens in* $16n(n-1)(n-2)(n-3)(n-4)/5!$ *hyper-planes.*

5. We are now in a position to extend the investigation to $(p+1)$ p-dimensional hyper-spheres, lying in a p-dimensional space. Generalizing from the preceding theorems we see that $(p+1)p/2$ of the $(p+1)p$ centers of similitude should lie in a $(p-1)$-dimensional space. Further, there will be 2^p such spaces. Hence,

(*K*) *The* $(p+1)p$ *centers of similitude of* $(p+1)$ p*-dimensional hyper-spheres lie, in general, in* 2^p $(p-1)$*-dimensional spaces containing* $(p+1)p/2!$ *centers of similitude.*

Next, consider n p-dimensional hyper-spheres, where $n \geqq p+1$. We have

(*L*) *The* $n(n-1)$ *centers of similitude of* n p*-dimensional hyper-spheres lie,*

in general, in 2^p $n(n-1)\cdots(n-p)/(p+1)!$ $(p-1)$*-dimensional hyper-planes each containing* $(p+1)p/2!$ *centers of similitude.*

Finally, we may slightly generalize this result and arrive at

(*M*) *The* $n(n-1)$ *centers of similitude of* n *m-dimensional hyper-spheres lie, in general, in* 2^p $n(n-1)\cdots(n-p)/(p+1)!$ $(p-1)$*-dimensional hyper-planes containing* $(p+1)p/2!$ *centers of similitude,* where $n > m \geqq p$.

6. For $(p+1)$ p-dimensional hyper-spheres there exists an interesting relation with regard to the number of exterior and interior centers in each $(p-1)$-dimensional hyper-plane.

We saw that for $p = 2$, there was one line containing 3 external, and 3 lines containing 1 external and 2 internal centers.

Monge showed for $p = 3$, that 1 plane contained 6 external centers, and 4 planes 3 external and 3 internal centers. The three which he omitted contain 2 external and 4 internal centers.

The results for $p = 4, 5, \cdots$ up to 10, may be readily obtained, and are sufficient to indicate a general relation. A display in tabular form will simplify the presentation. In the first table is shown the number of hyper-planes (planes, lines); and in the second is indicated, in a corresponding place, the number of external, and of internal centers of similitude contained in each corresponding hyper-plane (plane, line).

TABLE I.

$p=2$	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1
3	4	5	6	7	8	9	10	11
	3	10	15	21	28	36	45	55
			10	35	56	84	120	165
					35	126	210	330
							126	462

TABLE II.

$p=2$	3	4	5	6	7	8	9	10
3, 0	6, 0	10, 0	15, 0	21, 0	28, 0	36, 0	45, 0	55, 0
1, 2	3, 3	6, 4	10, 5	15, 6	21, 7	28, 8	36, 9	45, 10
	2, 4	4, 6	7, 8	11, 10	16, 12	22, 14	29, 16	37, 18
			6, 9	9, 12	13, 15	18, 18	24, 21	31, 24
					12, 16	16, 20	21, 24	27, 28
							20, 25	25, 30

It will be observed that in the first table the numbers, with the exception of the last in the columns where p is odd, are the regular coefficients in the expansion of $(a+b)^{p+1}$; the last numbers, when p is odd, are half the middle coefficients of the expansion. The composition of the second table is not especially complicated, but perhaps it will be worth while to append a series of formulæ for the general case.

When p *is odd:*

Number of $(p-1)$-dimensional hyperplanes	External Centers of Similitude	Internal Centers of Similitude
1	$\frac{p^2+p}{2}$	0
$p+1$	$\frac{p^2-p}{2}$	p
$\frac{(p+1)p}{2!}$	$\frac{p^2-3p+4}{2}$	$(p-1)2$
$\frac{(p+1)p(p-1)}{3!}$	$\frac{p^2-5p+12}{2}$	$(p-2)3$
$\vdots$	$\vdots$	$\vdots$
$\frac{(p+1)p(p-1)\cdots(p-k+2)}{k!}$	$\frac{p^2-(2k-1)p+(k-1)2k}{2}$	$(p-(k-1))k$
$\vdots$	$\vdots$	$\vdots$
$\frac{(p+1)p(p-1)\cdots\left(\frac{p+5}{2}\right)}{\left(\frac{p-1}{2}\right)!}$	$\frac{p^2-4p+3}{4}$	$\left(\frac{p+3}{2}\right)\left(\frac{p-1}{2}\right)$
$\frac{(p+1)p(p-1)\cdots\left(\frac{p+3}{2}\right)}{2\cdot\left(\frac{p+1}{2}\right)!}$	$\frac{p^2-1}{4}$	$\left(\frac{p+1}{2}\right)^2$

When p *is even,* the last two lines become:

$\frac{(p+1)p(p-1)\cdots\left(\frac{p+6}{2}\right)}{\left(\frac{p-2}{2}\right)!}$	$\frac{p^2+8}{4}$	$\left(\frac{p+4}{2}\right)\left(\frac{p-2}{2}\right)$
$\frac{(p+1)p(p-1)\cdots\left(\frac{p+4}{2}\right)}{\left(\frac{p}{2}\right)!}$	$\frac{p^2}{4}$	$\frac{p}{2}\left(\frac{p+2}{2}\right)$

HISTORICAL NOTE ON CENTERS OF SIMILITUDE OF CIRCLES.

By RAYMOND CLARE ARCHIBALD, Brown University.

In the paper on "Centers of Similitude of Circles and certain Theorems attributed to Monge. Were they known to the Greeks?"[1], I endeavored to show, especially through consideration of a problem in the book *On Tangencies*

[1] AMERICAN MATHEMATICAL MONTHLY, January, 1915, vol. 22, pp. 6–12. *Addenda:* In note 2, page 10, line 5, *for* cones *read* conics; on page 12, line 2, *for* side *read* sides. The two following historical notes may also be given:

(*a*) The theorem that "*The six centers of similitude of three coplanar circles lie by threes on four straight lines*" has been attributed to Monge who published it in 1798. Proof by analytical geometry that the three *external* centers of similitude are collinear was given by L. PUISSANT in his *Recueil de diverses propositions de géométrie*, Paris, 1801, pp. 50–56; it was probably here that the particular case of two of the circles being equal was first considered in recent times. No reference is made to Monge. There was a German edition by Hahn (Berlin, 1806). In the second French edition (Paris, 1809) mention is made (pp. 131–137) of Monge and the *general* theorem.

by Apollonius of Perga (about 225 B. C.), that several of the now well-known properties of centers of similitude of circles were also familiar to the Greeks. When writing the paper it did not occur to me to reinforce my argument by reference to another work by Apollonius, namely that *On Plane Loci*.[1] Pappus's account[2] of this work has been the basis of restorations and discussions by Fermat,[3] Schooten,[4] Simson,[5] Camerer,[6] Lhuilier,[7] Bonnycastle,[8] Breton (de Champ)[9] and Zeuthen.[10]

The *Plane Loci* consisted of two books which contained 147 propositions and figures, and 8 lemmas. Pappus states a general proposition, the detailed discussion of which he seems to indicate as the chief original contribution of Apollonius to the first book of his work.[11] This general proposition is as follows:

(*b*) To Monge (1798) is due the theorem (emended form of that given on p. 7 of the above mentioned paper): "*Given any four spheres in space, fixed in magnitude and position, and the six cones tangent to them in pairs, externally; then the six vertices lie in a plane and indeed on four straight lines in the plane.*" *If the six other tangent cones be drawn then the twelve conical vertices lie by sixes in five planes.* (This theorem is referred to by Mr. Brown in the preceding paper.) Monge overlooked the fact that there were three other planes with a similar property. This seems to have been first remarked by J. B. Durrande in *Annales de mathématiques* (Gergonne), Juillet, 1820, tome 11, pp. 18–20. He there calls the 8 planes "plans de similitude" and also gives synthetic proof of Monge's theorems for any positions of circles and spheres. The proofs of Monge are only applicable for circles and spheres external to one another. Durrande also used the term "axes de similitude" in connection with collinear centers of similitude of circles and spheres (pp. 10 and 17).

[1] The most recent historical sketch is by G. Loria in *Le Scienze esatte nell' antica Grecia*. Seconda edizione totalmente reviduta. Milano, 1914. Pp. 393–396 and 440–443. A reference may also be given to G. S. Klügel's *Mathematisches Wörterbuch*, erste Abtheilung, dritter Theil, Leipzig, 1808. Pp. 697–698.

[2] Pappi Alexandrini, *Collectionis*, edidit F. Hultsch, vol. 2, Berolini, 1877, pp. 660–671, 852–865. In C. I. Gerhardt's Greek-German edition, Halle, 1871, pp. 20–29; 166–175.

[3] *Varia opera mathematica*, D. Petri de Fermat. Tolosae, 1679, pp. 12–43; facsimile edition, Berolini, 1861—E. Brassine's *Précis des Oeuvres mathématiques de P. Fermat*, Toulouse, 1853, pp. 39–41—*Oeuvres de Fermat*, Paris, tome 1, 1891, pp. 3–51; tome 3, 1896, pp. 3–48. See also tome 2, pp. 5, 30, 56, 74, and 100.

[4] *Exercitationum mathematicarum. Liber III. Continens Apollonii Pergaei Loco Plana restituta.* Lugd. Batav. 1656, pp. 191–292—*Derde Bouck der mathematische Oeffeningen begrijpende, Apollonii Pergaei herstelde Vlacke Plaetsen . . .* door Franciscus Van Schooten. Amsterdam, 1660, pp. 185–272.

[5] *Apollonii Pergaei Locorum Planorum Libri II. Restituti* a Roberto Simson. Glasguæ, 1749. 252 pp.

[6] *Apollonius von Pergen ebene Oerter.* Wiederhergestellt von Robert Simson . . . Aus dem Lateinischen übersezt, mit . . . Aufgaben begleitet von J. W. Camerer. Leipzig, 1796. 455 pp. + 17 plates.

[7] *Éléments d'analyse géométrique et d'analyse algébrique appliquées a la recherche des lieux géométriques.* Par S. Lhuilier. Paris, 1809. "Lieux traités par Apollonius, suivant Simson" and "additions diverses aux lieux plans d'Apollonius," pp. 35–111.

[8] J. Bonnycastle, *Elements of Geometry.* Sixth edition, London, 1818. "De Locis Planis," pp. 371–375.

[9] *Recherches nouvelles sur les porismes d'Euclide.* Paris, 1855. Pp. 91–95. Also in *Journal de mathématiques pures et appliquées*, tome 20, 1855. Pp. 299–303.

[10] H. G. Zeuthen, *Die Lehre von den Kegelschnitten im Altertum.* Deutsche Ausgabe von R. v. Fischer-Benzon. Kopenhagen, 1886. Pp. 207–212.

[11] The early propositions of the *second* book include the following familiar results: (1) The locus of points, the difference of the squares of whose distances from two fixed points is constant, is a straight line perpendicular to the straight line joining the points; (2) The locus of the points,

"Two straight lines are drawn either from the same fixed point or from two fixed points, in the same direction or in such a way as to form a fixed angle; the lengths of these lines are in a constant ratio to one another or their rectangle is constant. If the extremity of one of them describes a plane locus given in position, the extremity of the second will also describe a plane locus, given in position, which is either of the same or of different species from the first."

A particular proposition included in this general one may be formulated as follows:

Through a point O draw lines $OP_1, OP_2, OP_3, \cdots$ to the various points $P_1, P_2, P_3, \cdots$ on a circle.[1] *Divide $OP_1, OP_2, OP_3, \cdots$ internally at $Q_1, Q_2, Q_3, \cdots$ respectively, and such that $OP_1 : OQ_1 = OP_2 : OQ_2 = \cdots = a$ const., and externally at $R_1, R_2, R_3, \cdots$ respectively, such that $OP_1 : OR_1 = OP_2 : OR_2 = \cdots = a$ const. Then the locus of the Q's is a circle,*[1] *and the locus of the R's is a circle.*[1]

Here O is the external center of similitude of the circles (P) and (Q), and the internal center of similitude of the circles (P) and (R). It is exactly such a proposition which Simson and others (*l. c.*) consider in their restorations in various cases when the circles (1) are exterior to one another; (2) intersect; (3) are such that one is inside of the other. The property of parallel radii joining corresponding points of the pairs of circles, arises in the course of the proofs.

My earlier argument that Apollonius was familiar with the centers of similitude of circles and some of their chief properties has thus been reinforced through consideration of another of his works.

A CIRCLE THEOREM.

By ROGER A. JOHNSON, Adelbert College, Western Reserve University.

THEOREM. *If three equal circles are drawn through a point, the circle through their other three intersections is equal to each of them.*

Proof. Denote the centers of the circles (see figure 1 of the next paper) by C_1, C_2, C_3, the intersections of C_2 and C_3 by O and P_1, those of C_3 and C_1 by O and P_2, those of C_1 and C_2 by O and P_3. Then $OC_2P_1C_3$ is a rhombus, and so is $OC_3P_2C_1$. Hence, C_2P_1 and C_1P_2 are equal and parallel, $C_1C_2P_1P_2$ is a parallelogram, and P_1P_2 is equal to C_1C_2. Thus the triangles $C_1C_2C_3$ and $P_1P_2P_3$ are congruent, and have equal circumcircles. But the circumcircle of the former has its center at O, and is equal to each of the given circles. Hence, the circle through P_1, P_2, P_3 is equal to each of the given circles.

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the ratio of whose distances from two fixed points is constant, is either a straight line or a circle —the Circle of Apollonius. Eutocius gives the proof of Apollonius for this latter locus (Apollonius, ed. Heiberg, Vol. 2, pp. 180–185). The name "Circle of Apollonius" is, however, a misnomer, since the construction of this locus connected with his name appears in exactly the same form at a much earlier date in Aristotle's *Meteorologica*, III, 5, 376 f.

[1] "Circle" is here considered as a curved line. The cases of this proposition when we substitute "straight line" for "circle" were discussed by Euclid in Propositions 35–36 of his *Data* (Simson's edition, Prop. 39—for example: *Elements of Elucid . . . also Euclid's Data*, 9th ed., Edinburgh, 1793, pp. 393–394).

COROLLARY. *Each of the four points O, P_1, P_2, P_3, is the orthocenter of the triangle of the other three, and the set of points has all the well-known properties of an orthocentric system.*

Singularly enough, this remarkable theorem appears to be new. A rather cursory search in several of the treatises on modern elementary geometry fails to disclose it, and the author has not yet found any person to whom it was known. On the other hand, the figure is so simple (especially as it can be drawn and the theorem verified with a coin or other circular object) that it seems almost out of the question that the fact can have escaped detection. Even if geometers have overlooked it, someone must have noticed it in casually drawing circles. But if this were the case, it seems like a theorem of sufficient interest to receive some prominence in the literature, and therefore ought to be well known. It is hoped that if any reader recognizes the theorem, or knows where it has already been given, he will report the same. Of course, the converse theorem that the four circumcircles of an orthocentric system are equal, is well known.

REMARKS ON THE FOREGOING CIRCLE THEOREM.

By ARNOLD EMCH, University of Illinois.

1. The foregoing theorem proved by Mr. Johnson gains additional interest in connection with the theory of circular inversion.

Before this fact is pointed out, another proof of the theorem, equally notable on account of its extreme simplicity, will be given. Using the same notation as Mr. Johnson, and denoting the given circles through O by $\alpha_1, \alpha_2, \alpha_3$, as shown[1] in Fig. 1, we have $\measuredangle OP_3P_2 = \measuredangle OP_1P_2$, because the circles α_1 and α_3 are equal and have the common chord OP_2 subtending those angles. Likewise, $\measuredangle OP_2P_3 = \measuredangle OP_1P_3$. Consequently $\measuredangle OP_3P_2 + \measuredangle OP_2P_3 = \measuredangle OP_1P_2 + \measuredangle OP_1P_3 = \measuredangle P_2P_1P_3$, so that $\measuredangle P_2OP_3$ and $\measuredangle P_2P_1P_3$ are supplementary. But also $\measuredangle P_2AP_3$ and $\measuredangle P_2OP_3$ are supplementary (A is any point on α_1); hence $\measuredangle P_2P_1P_3 = \measuredangle P_2AP_3$. From this follows immediately that the circle α_4 through $P_1P_2P_3$ is equal to the circle α_1, and consequently to α_2 and α_3. As the sum of the six angles in the equality

$$\measuredangle OP_2P_1 + \measuredangle OP_2P_3 + \measuredangle OP_3P_2 = \measuredangle OP_3P_1 + \measuredangle OP_1P_3 + \measuredangle OP_1P_2$$

is equal to the sum of the angles in the triangle $P_1P_2P_3$, the left and right hand sides are equal to a right angle, *i. e.*, P_2P_3Q is a right triangle, and consequently P_3Q is perpendicular to P_1P_2. Likewise P_2O and P_1O prolonged are perpendiculars to P_1P_3 and P_3P_2, respectively; and O is the orthocenter of the triangle $P_1P_2P_3$. In a similar manner it can be shown, that P_1, P_2, P_3, are orthocenters of the corresponding triangles OP_2P_3, OP_3P_1, OP_1P_2.

[1] For figures in which O is without the triangle $P_1P_2P_3$, the proposition can be proved in a similar manner by angular relations.

2. Consider in Fig. 2 any triangle $P_1'P_2'P_3'$ with its circumscribed circle α_4' and the inscribed circle I with center O. Denote the sides of the triangle respectively by α_1', α_2', α_3', and invert the whole figure with respect to I as the circle of inversion. Let Q_1, Q_2, Q_3 be the points of tangency of the inscribed circle with the sides of the triangle; then α_1', α_2', α_3', supposed to be indefinitely extended,

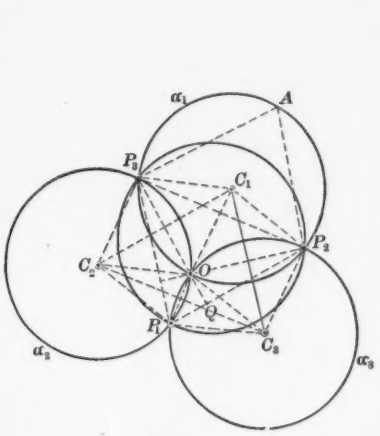


Fig.1

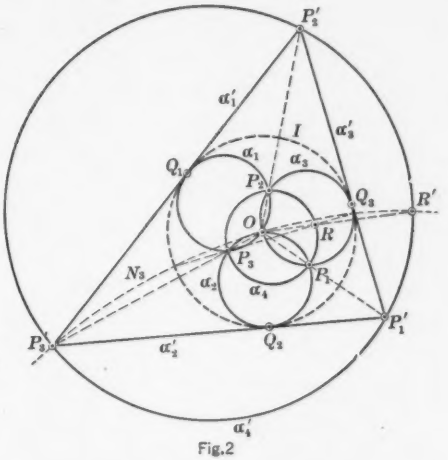


Fig.2

are inverted into three circles α_1, α_2, α_3, which pass through O and touch I at Q_1, Q_2, Q_3, respectively. They intersect in three points P_1, P_2, P_3 which in the same order are the inverse of P_1', P_2', P_3', and are evidently equal circles. The circle α_4 through P_1, P_2, P_3 is the inverse of the circumscribed circle α_4' and, according to the circle theorem, is equal to α_1, α_2, α_3. We have therefore the

THEOREM: *The indefinitely extended sides of a triangle and its circumscribed circle are inverted into four equal circles, with the inscribed (or escribed) circle of the triangle as the circle of inversion.*

Conversely, the figure of the circle theorem (Fig. 1) may always be inverted into a triangle with its in- (or escribed) and circumscribed circles.

3. Among the number of other propositions which may be derived from this theorem I shall prove only one.

Through any of the vertices, say P_3', of the triangle (Fig. 2) and the point O pass a circle N_3 cutting α_4' orthogonally, and let R' be the other point of intersection of N_3 with α_4'. The question is, what relation does the point R' have with respect to the triangle? Invert N_3 with I as the circle of inversion. The inverse is a straight line P_3R which intersects α_4 orthogonally. P_3R is therefore a diameter of α_4, and R is therefore the point of tangency of the circle having P_3 as a center and touching each of the three equal circles α_4, α_1, α_2. Inverting back, we find, that R' is the point of tangency of a circle which is tangent to the

circumscribed circle of the triangle and the two sides of the triangle that meet at P_3'. A similar result is obtained by passing circles through P_1', P_2' and O, orthogonal to α_4'. The result may be stated as the

THEOREM: *A circle which passes through a given vertex of a triangle and the center of the inscribed circle, and is orthogonal to the circumscribed circle, cuts the latter in another point which is the point of tangency of a circle that is tangent to the circumscribed circle and to the two sides of the triangle meeting at the given vertex.*

KANSAS SECTION OF THE MATHEMATICAL ASSOCIATION OF AMERICA.

The first meeting of the Kansas Section of the Mathematical Association of America was held at Lawrence, Kansas, on Saturday, March 18, 1916, in room 105, Administration Building, University of Kansas.

Following are the names of those present, together with the institutions represented:

University of Kansas: J. N. VAN DER VRIES, C. H. ASHTON, U. G. MITCHELL, H. E. JORDAN, J. J. WHEELER, S. LEFSCHETZ, A. W. LARSEN, K. L. HOLZINGER, L. L. STEIMLEY, C. A. NELSON, J. M. JACOBS, P. W. HARNLEY.

Kansas State Agricultural College: B. L. REMICK, A. E. WHITE, W. T. STRATTON, H. E. PORTER.

Kansas State Normal: THEODORE LINDQUIST.

Fairmount College: A. J. HOARE.

College of Emporia: T. E. MERGENDAHL.

Washburn College: W. A. HARSHBARGER, MARY NEWSON.

Baker University: W. H. GARRETT.

McPherson College: A. B. FRIZELL.

Bethel College: D. H. RICHERT.

Friends University: O. W. DUEKER.

Campbell College: T. L. BOUSE.

Kansas City, Kan., High School: ELIZABETH G. FLAGG, EMMA HYDE, and LUCY DOUGHERTY.

This was the second meeting of an association formed in 1915 for the improvement of the teaching of collegiate mathematics in the state of Kansas.

The Kansas Association has now become a section of the national organization, The MATHEMATICAL ASSOCIATION OF AMERICA, founded at Columbus, Ohio, December 30, 1915.

The meeting was called to order by Professor A. J. Hoare, of Fairmount College.

Professor U. G. Mitchell, of the University of Kansas, gave a report on the organization meeting of the National Association at Columbus, Ohio. Professor Mitchell was the delegate of the Kansas Section, which was the first body to make application for admission as a section of the ASSOCIATION.

MEMBERS OF THE KANSAS SECTION OF THE MATHEMATICAL ASSOCIATION OF AMERICA, IN SESSION AT LAWRENCE, KANSAS, MARCH 18, 1916.

Professor J. N. Van der Vries, of the University of Kansas, gave a paper on "Geometry for College Juniors and Seniors." The discussion of this paper was led by Professor Mary Newson, of Washburn College. The paper will be published in a later issue of the MONTHLY.

The committee in charge consisted of Professor A. J. Hoare, *Chairman*, Dr. S. Lefschetz, Professor B. L. Remick, Professor W. A. Harshbarger, and Professor T. E. Mergendahl, *Secretary*.

The next meeting will be held at the time of the State teachers' association in November, 1916. This meeting will be devoted to a consideration of a rearrangement of the freshman college algebra work, due to a reduction by legislative enactment of one-half unit in the requirement for admission.

The meeting next spring will be devoted to a consideration of the algebra course for college juniors and seniors, to be arranged in accordance with the freshman work decided upon at the fall meeting.

T. E. MERGENDAHL,
Secretary of the Kansas Section.

A TRIBUTE TO ANDREW WHEELER PHILLIPS.

In the death of Professor Andrew Wheeler Phillips, January 20, 1915, Yale University lost one of her truly great men, and without doubt the most beloved teacher ever connected with that institution.

Born in 1844 amid the rocks and hills of Connecticut, he passed his early life as a typical New England farmer boy. His home training was under a father and mother thrifty, intelligent, and devoutly religious. The district school and three summer vacations spent in a select school at Jewett City constituted his elementary schooling. His liking for mathematics came early, for in a letter he wrote: "When I got bigger and was trusted to go off alone and harrow a field, how many times have I let the oxen rest under the shade of a big tree as I smoothed off the rough ground and drew geometrical diagrams and solved knotty problems, while the oxen looked at me with tender eyes and chewed their cuds with contentment and happiness!"

He taught his first school when a lad of sixteen, and at the age of twenty he became a teacher in the Cheshire School. Though he had never been a student in an academy or high school, he soon ranked with the best teachers in the academy at Cheshire. During the forty-seven years, from his appointment at Cheshire until his retirement from the Yale Graduate School in 1911, he did every year full work as a teacher and administrative officer. This shows that the education which fitted him so well for his life work was largely the product of his own private study and keen observation. Professor Newton admitted him to his class in mathematics for graduates as a special student in the fall of 1870, and placed the recitations on Saturday so that Phillips might attend. After three years he received the degree of Bachelor of Philosophy, and in the winter

of 1876 he left the academy at Cheshire to become a tutor in mathematics at Yale. He continued his studies in mathematics and in other departments, and his advancement to the Doctor's degree and to higher appointments was steady and natural. Anyone who never knew him might ask how a man who never was an undergraduate student for a day in a college, or even a pupil in a high school, should become the head of the Yale Graduate School.

Professor Phillips had much technical mathematical ability and wrote several books and numerous articles for the mathematical journals. It is a significant fact that each one of his books was written in collaboration with one of his colleagues. Dean Henry P. Wright said of him: "He lived to do good and make others happy, and could do nothing to lessen the happiness even of those who had injured him. He was wholly unselfish. I cannot think of him as forming any plan solely for his own advancement. He seemed to ignore his own interests in his devotion to the interests of others. He was always busily occupied, but it was for other persons or for other objects than his own—for the Cheshire school, or the Hopkins grammar school, or the Hotchkiss school; for the college; for the graduate department; for the Bicentennial Fund; for St. Thomas's church in New Haven; for his students, and those who had been his students; for his colleagues; for his friends, or his family; for the assistants in his office, or for the servants in his house. He was always ready to help when an opportunity offered."

As a teacher of young men, he was firm and kind. Every student who came in contact with Andy, as we used to call him, immediately felt his wonderful personality. His love and enthusiasm for work were so great that even the laziest student somehow managed to work some for him. One of them once said: "Work is catching in Andy's class." His interest in his students was so great that every student who took any course with him felt that he was an intimate and personal friend. His memory is now dear to all his students, not for the mathematics he taught so well, but for the high ideals he inculcated by his beautiful character and example. His students and friends will always think of him as they knew him, full of life, sunshine, and human sympathy.

H. T. BURGESS.

UNIVERSITY OF WISCONSIN.

BOOK REVIEWS.

Send all communications to W. H. BUSSEY, University of Minnesota.

Diophantine Analysis. By ROBERT D. CARMICHAEL. John Wiley and Sons, Inc., New York, 1915. vi+118 pages. $1.25.

Greek theory of numbers, like Greek geometry, has come down to us in a remarkably disconnected and unsystematized form. The method used in solving one problem gives little hint of the method to be used in solving a second which may be apparently closely related to the first. The theorems in Euclid are

arranged in masterly sequence, but the proofs stand by themselves, a different scheme for each. In Diophantine analysis, not only do the solutions stand apart from each other, but the theorems themselves seem unrelated and patchy. This characteristic has made it the favorite hunting ground for the amateur, who finds there little necessity for relating his work to other theories with which he may not be on speaking terms. This same characteristic has also attracted to the subject some of the finest minds in the world of science: men who have labored to pick out of this vast, confused heap of unrelated materials some connecting thread with which to bind the whole theory together. No one thing indicates more clearly the miraculous insight of Fermat than his instinctive appreciation of the importance of the so-called Pellian equation. Whether he had the key to his famous last theorem or not, he certainly had some clear notion of the place it should occupy in the Theory of Numbers. Fermat tried to do for Diophantine Analysis what his great contemporary, Desargues, did for pure geometry. Since Fermat's discovery of the method of "infinite descent" very few general methods have been found applicable to this kind of analysis. Since all solutions of the Pellian equation seem to be only disguises of the solution by continued fractions, it is not improbable that Fermat was familiar with this powerful algorithm.

Professor Carmichael in his book calls attention to two other methods which he calls the "method of multiplicative domains," and the "method of functional equations." As an illustration of what he means by the "method of multiplicative domains" he derives from the identity:

$$(m^2 + n^2)^3 = (m^3 - 3mn^2)^2 + (3m^2n - n^3)^2$$

the following two double-parameter solutions of the equation,

$$x^2 + y^2 = z^3:$$

$$x = m^3 + mn^2, \qquad y = m^2n + n^3, \qquad z = m^2 + n^2;$$

$$x = m^3 - 3mn^2, \qquad y = 3m^2n - n^3, \qquad z = m^2 + n^2.$$

The theory of the "multiplicative domain" is developed at some length in Chapter II. The above illustration indicates what may be deduced from the fact that the sum of two squares multiplied by the sum of two squares is again the sum of two squares. This general multiplicative property does not hold for the sum of two cubes, and Professor Carmichael shows how by increasing the number of variables a new set of numbers may be obtained which includes the given set and enjoys the multiplicative property. Thus, instead of the numbers represented by $x^3 + y^3$, which do not form a multiplicative domain, he uses the numbers $x^3 + y^3 + z^3 - 3xyz$, which do form such a domain. The underlying theory is contained in the theory of composition of forms. Thus the theorem that "No form can be transformed into the product of two forms of the same sort (irreducible forms being understood to have at least one invariant different from zero), unless the number of its indeterminates is a multiple of its order," indicates why the numbers $x^3 + y^3$ do not form a multiplicative domain.

In the "method of functional equations" Professor Carmichael makes use of rational solutions of certain types of Diophantine problems of importance in the work of Diophantus itself. A general systematic development of this method is not attempted.

Chapters III and IV, on the equations of the third and fourth degrees, are very valuable in giving in elegant form the principal results so far obtained in this field. Professor Carmichael has done good service in collecting and arranging not only his own researches but the results of the labor of others as well. A short account is given in Chapter V of certain higher equations, ending with a very complete statement of our knowledge regarding Fermat's Last Theorem.

Professor Carmichael has omitted entirely the theory of binary quadratic forms and has made no use of the theory of continued fractions. This has resulted in some rather unsightly patches in the book. The discussion of the equation $x^2 - Dy^2 = z^2$ makes the author almost as much trouble, and takes up almost as much space as a straightforward discussion of the continued fraction representing a quadratic surd would require.

The book will serve a valuable end in stimulating interest in this delightful branch of mathematics. There are many interesting and suggestive problems at the end of each chapter, some of which are excellent material for further research.

D. N. LEHMER.

UNIVERSITY OF CALIFORNIA.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND R. P. BAKER.

[Send all Communications to B. F. FINKEL, Springfield, Mo.]

PROBLEMS FOR SOLUTION.

ALGEBRA.

457. Proposed by FRANK IRWIN, University of California.

If a be any number prime to m and m/a be developed as a continued fraction,

$$\frac{m}{a} = a_1 + \frac{1}{a_2} + \frac{1}{a_3} + \cdots + \frac{1}{a_{k-1}} + \frac{1}{a_k},$$

with $a_1 \neq 0$, then there will exist a number b such that $m/b = a_k + 1/a_{k-1} + \cdots + 1/a_2 + 1/a_1$. Show that $ab \equiv \pm 1 \pmod{m}$ and determine the sign.

458. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

Show that n terms of the series $1 + 3 + 4 + 6 + 7 + 9 + 10 + \cdots$ is $\frac{1}{4}(n+1)(3n-1)$ when n is odd, and $n/2[(3n/2) + 1]$ when n is even.

459. Proposed by C. N. SCHMALL, New York City.

By d'Alembert's test, or otherwise, show that in the infinite series

$$x + \frac{2^2x^2}{2!} + \frac{3^3x^3}{3!} + \frac{4^4x^4}{4!} + \cdots + \frac{n^nx^n}{n!} + \cdots$$

the upper limit of the interval of convergence is $1/e$ where e is the Naperian base, *i. e.*, $x < 1/e$ when the series is convergent. (Bromwich's *Infinite Series*, pp. 28, 33.)

GEOMETRY.

489. Proposed by NATHAN ALTSHILLER, The University of Colorado.

The parallels to the asymptotes a, b of a given hyperbola, drawn from a variable point of the curve, meet a and b in P, Q respectively. The line PQ envelops an hyperbola whose asymptotes are a and b.

490. Proposed by ELMER E. MOOTS, University of Arizona.

In any quadrilateral $ABCD$, let AC and BD be the diagonals intersecting in K. On AC, lay off CR equal to AK. Join B and R. Connect the middle point G of BR with D. On GD, lay off GM equal to $\frac{1}{4}GD$. Show that M is the center of gravity of the quadrilateral.

491. Proposed by N. P. PANDYA, Sojitra, India.

In a triangle $mx = b$ and $nx = c$, determine a relation between m, n, x, A and s and solve it for x.

CALCULUS.

407. Proposed by PAUL CAPRON, Annapolis, Maryland.

A coffee pot in the form of a conical frustum, 10 inches high, with a lower base 8 inches in diameter and an upper base 6 inches in diameter, is held on a slant so that the lower base is barely covered by the coffee within, and the upper base is barely uncovered. How much coffee does the pot contain?

408. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

The ellipse $(x^2/81) + (y^2/16) = 1$ is revolved around the y-axis. Find the area of the surface generated.

409. Proposed by B. J. BROWN, Victor, Colorado.

Integrate the equation

$$\frac{\partial^2 z}{\partial x \partial y} + \frac{1}{x+y}\left(\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\right) - \frac{2}{(x+y)^2} z = 0.$$

MECHANICS.

326. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

A uniform beam, of length $2l$, rests in equilibrium against a smooth vertical wall and upon a peg at a distance a from the wall; show that the inclination of the beam to the vertical is

$$\sin^{-1}\left(\frac{a}{l}\right)^{\frac{1}{2}}.$$

327. Proposed by C. N. SCHMALL, New York City.

An inclined plane makes an angle ϕ with the horizontal plane, and from its foot a body is projected upward at an angle ψ to the plane, and with velocity v. Show that it will strike the plane *perpendicularly* if $\tan \psi = \frac{1}{3} \cot \phi$ and that its range up the plane in that case will be

$$\frac{2v^2 \sin \phi}{g(1 + 3 \sin^2 \phi)}.$$

NUMBER THEORY.

244. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

Determine the rational value of x that will render $x^2 + px + q$ a perfect square. What value of x will render $x^2 - 7x + 2$ a perfect square?

245. Proposed by NORMAN ANNING, Chilliwack, B. C.

When all the letters denote positive integers and when the a's are primes of the form $4k + 1$, the equation

$$x^2 + y^2 = (a_1 a_2 a_3 \cdots a_m)^n,$$

has, in Legendre's notation, $E((n+1)^m/2)$ solutions. Show that in 2^{m-1} of these solutions x and y are relatively prime.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

445. Proposed by S. A. JOFFE, New York City.

Sum the series

$$\binom{cn}{a} - \binom{a}{1}\binom{cn-c}{a} + \binom{a}{2}\binom{cn-2c}{a} - \cdots + (-1)^a \binom{cn-ca}{a}.$$

I. Solution by the Proposer.

This is a generalization of the series forming the first member of equation (2) in the Proposer's solution of problem No. 424 (*June* 1915 issue, p. 205), obtained by multiplying the upper indices $n, n-1, n-2, \cdots$ by the constant factor c. Following the method employed in that solution, we find that the given series equals

$$\Delta^a \binom{cn-ca}{a},$$

the finite differences being taken with respect to n.

Now

$$\Delta_x \binom{cx}{a} = \binom{c\overline{x+1}}{a} - \binom{cx}{a},$$

the second member of which may be written in the following form:

$$\binom{cx+c}{a} - \binom{cx+c-1}{a} + \binom{cx+c-1}{a} - \binom{cx+c-2}{a} + \binom{cx+c-2}{a} - \cdots + \binom{cx+1}{a} - \binom{cx}{a},$$

since all these terms, except the first and last, alternately cancel each other. Combining the terms in pairs and noticing that

$$\binom{cx+c}{a} - \binom{cx+c-1}{a} = \binom{cx+c-1}{a-1}, \quad \binom{cx+c-1}{a} - \binom{cx+c-2}{a} = \binom{cx+c-2}{a-1},$$

etc., we have

$$\Delta_x \binom{cx}{a} = \binom{cx+c-1}{a-1} + \binom{cx+c-2}{a-1} + \cdots + \binom{cx+1}{a-1} + \binom{cx}{a-1},$$

which means that the first difference, taken with respect to x, of the binomial coefficient $\binom{cx}{a}$ having for its lower index a, equals the sum of c binomial coefficients, each having for its lower index $a-1$.

In the same manner, the second difference $\Delta_x^2 \binom{cx}{a}$ may be expressed as the sum of $c \cdot c = c^2$ binomial coefficients, each having $a-2$ for its lower index; and continuing this process, we find that the ath difference $\Delta_x^a \binom{cx}{a}$ may be expressed as the sum of c^a binomial coefficients, each having for its lower index $a - a = 0$ and hence each equal to 1. In other words,

$$\Delta_x^a \binom{cx}{a} = c^a,$$

and, similarly,

$$\Delta_n^a \binom{c\overline{n-a}}{a} = c^a.$$

We thus arrive at the result:

$$\binom{cn}{a} - \binom{a}{1}\binom{cn-c}{a} + \binom{a}{2}\binom{cn-2c}{a} - \cdots + (-1)^a \binom{cn-ca}{a} = c^a.$$

II. Solution by Norman Anning, Chilliwack, B. C.

Define operators E and Δ as follows:

$$E^h f(n) = f(n+h), \qquad \Delta f(n) = f(n+1) - f(n) = (E-1)f(n).$$

Then the given expression,

$$\binom{cn}{a} - \binom{a}{1}\binom{cn-c}{a} + \binom{a}{2}\binom{cn-2c}{a} - \cdots + (-1)^a \binom{cn-ca}{a}$$

$$= (1 - E^{-1})^a \binom{cn}{a} = (\Delta E^{-1})^a \binom{cn}{a} = \Delta^a E^{-a} \binom{cn}{a} = \Delta^a \binom{cn-ca}{a}.$$

Now

$$\binom{cn-ca}{a} = \frac{1}{a!}\{c^a n^a + \text{terms in } n \text{ of degree lower than } a\}.$$

Hence,

$$\Delta^a \binom{cn-ca}{a} = \frac{1}{a!}\{c^a \cdot a! + 0\} = c^a.$$

446. Proposed by Clifford N. Mills, Brookings, South Dakota.

Solve the equations

$$x^2(y-z) = l^2(m-n), \qquad y^2(z-x) = m^2(n-l), \qquad z^2(x-y) = n^2(l-m).$$

Solution by H. S. Uhler, Yale University.

An equation involving only one variable x may be obtained in the following manner. Equating the expressions for z derived from the first two given equations we find

$$y - l^2(m-n)/x^2 = x + m^2(n-l)/y^2$$

or

$$x^2y^2 - x^2y^3 + m^2(n-l)x^2 + l^2(m-n)y^2 = 0. \tag{1}$$

Substituting the value of z from the first in the third given equation and reducing, we obtain

$$x^5y^3 - x^4y^3 - n^2(l-m)x^4 - 2l^2(m-n)x^3y + 2l^2(m-n)x^2y^2 + l^4(m-n)^2x - l^4(m-n)^2y = 0. \tag{2}$$

Multiplying (1) by x^3 and subtracting (2) from the product gives, after removing the factor $m-n$,

$$(lm - mn + nl)x^4 - 2l^2x^3y + l^2x^3y^2 + l^4(m-n)x - l^4(m-n)y = 0. \tag{3}$$

Multiplying (1) by l^2, (3) by y, adding, and dividing by x,

$$(lm - mn + nl)x^3y - l^2x^2y^2 + l^2m^2(n-l)x + l^4(m-n)y = 0. \tag{4}$$

Adding (3) and (4) and dividing by x,

$$(lm - mn + nl)x^3 + (lm - mn + nl - 2l^2)x^2y + l^2[l^2(m-n) + m^2(n-l)] = 0. \tag{5}$$

Substituting the expression for y from (5) in (4) and removing the factors $(l-m)(l-n)$ we derive the following quadratic in x^3, namely

$$(lm - mn + nl)^2x^6 - 2l^4[m^2(l-n) + n^2(l-m)]x^3 - l^8(mn - nl + lm)(nl - lm + mn) = 0.$$

Consequently

$$x^3 = l^3$$

and

$$x^3 = \frac{l^3[l^2(m-n)^2 - m^2n^2]}{(lm - mn + nl)^2}.$$

Finally, the six values of x are

$$l,\ l\omega,\ l\omega^2,\ \frac{l[l^2(m-n)^2-m^2n^2]^{\frac{1}{3}}}{(lm-mn+nl)^{\frac{2}{3}}},\ \frac{l[l^2(m-n)^2-m^2n^2]^{\frac{1}{3}}\omega}{(lm-mn+nl)^{\frac{2}{3}}},\ \frac{l[l^2(m-n)^2-m^2n^2]^{\frac{1}{3}}\omega^2}{(lm-mn+nl)^{\frac{2}{3}}},$$

where

$$\omega \equiv (-1+i\sqrt{3})/2.$$

The corresponding values of y and z may be obtained at once by cyclical permutation of the letters l, m, and n.

Remarks: The above analysis was verified by direct substitution of the six sets of values in the given equations. The elimination of y between equations (1) and (2) can be performed in a more elegant manner by Sylvester's dialytic method, but the simplification of the determinant of the eighth order requires too much space for publication.

Also solved by A. H. HOLMES and NORMAN ANNING.

GEOMETRY.

471. Proposed by C. N. SCHMALL, New York City.

In the ellipse $x^2/a^2+y^2/b^2=1$, an equilateral hexagon is inscribed with two sides parallel to the major axis. In the major auxiliary circle the same thing is done. If H_1 and H_2 be the sides of the hexagons, and e the eccentricity of the ellipse, show that $H_1 : H_2 :: 4-2e^2 : 4-e^2$.

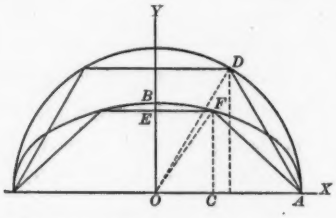
SOLUTION BY J. A. CAPARO, University of Notre Dame.

Since AD is the side of a regular hexagon inscribed in the major circle of the ellipse $x^2/a^2+y^2/b^2=1$, we have

$$DA = OA = a = H_2$$

and since

$$e^2=\frac{a^2-b^2}{a^2}, \qquad b^2=H_2^2(1-e^2), \qquad \therefore\ \frac{x^2}{H_2^2}+\frac{y^2}{H_2^2(1-e^2)}=1. \tag{1}$$



Now $EF = H_1/2$; therefore, from (1) we have

$$FC^2 = y^2 = \frac{(4H_2^2-H_1^2)(1-e^2)}{4}.$$

Also $CA = OA - OC$. Hence, $CA = H_2 - (H_1/2)$; and since $FA^2 = FC^2 + CA^2$, we have

$$4H_1^2 = (4H_2^2-H_1^2)(1-e^2)+(2H_2-H_1)^2.$$

Write $H_1/H_2 = x$. Then, substituting and reducing, we have

$$x^2(4-e^2)+4x+4(e^2-2)=0.$$

Solving for x, we have

$$x=\frac{-2\pm 2(e^2-3)}{4-e^2}.$$

Using the negative sign, we have $x = (4-2e^2)/(4-e^2)$; and since $x = H_1/H_2$,

$$H_1 : H_2 :: 4-2e^2 : 4-e^2.$$

Also solved by C. E. HORNE, FRANK IRWIN, HAROLD T. DAVIS, NORMAN ANNING, ELIJAH SWIFT, and C. N. SCHMALL.

473. Proposed by FRANK R. MORRIS, Gendale, Calif.

What is the length of the longest rectangle an inch wide that can be placed inside another rectangle 12 inches long and 8 inches wide? Obtain the result correct to the third decimal.

SOLUTION BY WILLIAM W. JOHNSON, Cleveland, Ohio.

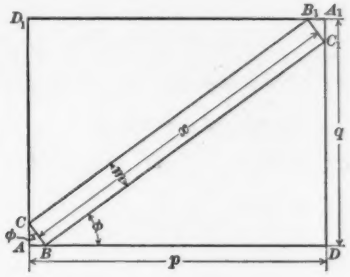
Using the notation of the figure, we have $AB = m \sin\phi$, $BD = x\cos\phi$ and $AB + BD = p$. Then,

$$m\sin\phi + x\cos\phi = p. \tag{1}$$

$AC = m\cos\phi$, $CD_1 = DC_1 = x\sin\phi$ and $AC + CD_1 = q$. Then,

$$x\sin\phi + m\cos\phi = q. \tag{2}$$

Solving (1) and (2) for $\sin\phi$ and $\cos\phi$, we get



$$\sin\phi = \frac{pm - qx}{m^2 - x^2}, \qquad \text{and} \qquad \cos\phi = \frac{qm - px}{m^2 - x^2}.$$

Squaring, adding, and putting $\sin^2\phi + \cos^2\phi = 1$, we get

$$\left[\frac{pm - qx}{m^2 - x^2}\right]^2 + \left[\frac{qm - px}{m^2 - x^2}\right]^2 = 1.$$

Expanding, clearing of fractions and reducing we obtain

$$x^4 - (p^2 + q^2 + 2m^2)x^2 + 4pmqx - (p^2 + q^2 - m^2)m^2 = 0.^1$$

Putting, $p = 12$, $q = 8$, and $m = 1$, we get

$$x^4 - 210x^2 + 84x - 207 = 0.$$

Solving this equation by Horner's method, we find $x = 13.5176$. Therefore, the length of the inscribed rectangle is 13.5176 inches.

Also solved by H. C. FEEMSTER and J. A. CAPARO.

CALCULUS.

390. Proposed by WILSON L. MISER, University of Arkansas.

Show that the triangle whose area is a constant and whose perimeter is a minimum is equilateral.

SOLUTION BY J. A. CAPARO, University of Notre Dame.

Let ABC be the given triangle of constant area equal to k. Denote the base AB of the triangle by x, the perpendicular CD upon AB by y, the perimeter by P and the angle CAB by α.

[1] See Merriman's *The Solution of Equations*, p. 20, 4th ed.

Then,

$$k = \frac{xy}{2}. \tag{1}$$

Now, $P = AC + CB + AB$, and for a minimum $dP/dy = 0$ and $\delta P/\delta\alpha = 0$. But, $AC = y/\sin\alpha = y\csc\alpha$; and since $\overline{CB}^2 = \overline{CD}^2 + \overline{DB}^2$ and $DB = AB - AD = x - y\cot\alpha$,

$$CB = [y^2 + (x - y\cot\alpha)^2]^{\frac{1}{2}} = \left(y^2\csc^2\alpha + \frac{4k^2}{y^2} - 4k\cot\alpha\right)^{\frac{1}{2}}.$$

Then

$$P = y\csc\alpha + \left(y^2\csc^2\alpha + \frac{4k^2}{y^2} - 4k\cot\alpha\right)^{\frac{1}{2}} + \frac{2k}{y}.$$

$$\therefore\ \frac{dP}{dy} = \csc\alpha + \frac{1}{2}\left(y^2\csc^2\alpha + \frac{4k^2}{y^2} - 4k\cot\alpha\right)^{-\frac{1}{2}}\left(2y\csc^2\alpha - \frac{8k^2}{y^3}\right) - \frac{2k}{y^2} = 0,$$

which can be written

$$(y^2\csc\alpha - 2k)[(y^4\csc^2\alpha + 4k^2 - 4ky^2\cot\alpha)^{\frac{1}{2}} - (2k + y^2\csc\alpha) = 0.$$

Hence,

$$y^2\csc\alpha - 2k = 0 \tag{2}$$

and

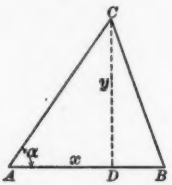
$$y^4\csc^2\alpha + 4k^2 - 4ky^2\cot\alpha = (2k + y^2\csc\alpha)^2,$$

which reduces to

$$-4ky^2\cot\alpha = 4ky^2\csc\alpha.$$

Hence,

$$-\cot\alpha = \csc\alpha \qquad \text{or} \qquad \alpha = 180°.$$



Substituting (1) in (2) we have $y^2\csc\alpha = xy$ or $\sin\alpha = y/x$; and since $\sin\alpha = y/AC$, $AC = x$, that is, $AC = AB$. Also

$$dP/d\alpha = -y\csc\alpha\cot\alpha + \tfrac{1}{2}(y^2\csc\alpha + 4k^2/y^2 - 4k\cot\alpha)^{-\frac{1}{2}}(-2y^2\csc^2\alpha\cot\alpha + 4k\csc^2\alpha) = 0;$$

or reducing,

$$\cos\alpha\cdot(y^4\csc^2\alpha + 4k^2 - 4ky^2\cot\alpha)^{\frac{1}{2}} = 2k - y^2\cot\alpha.$$

Squaring both sides and reducing, we get

$$(y^2\cot\alpha - k)(1 - \cos^2\alpha) = 0.$$

Hence,

$$1 - \cos^2\alpha = 0 \qquad \text{and} \qquad y^2\cot\alpha - k = 0. \tag{3}$$

Solving (2) and (3) for α, we get $\tan\alpha = 2\sin\alpha$. Hence $\alpha = 60°$; and since we have shown above that $AB = AC$ and $\alpha = 60°$ is the included angle between these two lines, it follows that $CB = AB = AC$.

Also solved by FRANK IRWIN.

391. Proposed by H. B. PHILLIPS, Massachusetts Institute of Technology.

If $0 < \lambda < 1$ and $0 < x < \pi$ show that the function $(\sin\lambda x)/(\sin x)$ increases as x increases.

I. SOLUTION BY H. S. UHLER, Yale University.

Applying the well-known formula

$$\sin\theta = \theta\left[1 - \left(\frac{\theta}{\pi}\right)^2\right]\left[1 - \left(\frac{\theta}{2\pi}\right)^2\right]\left[1 - \left(\frac{\theta}{3\pi}\right)^2\right]\cdots, \qquad [\theta^2 < \infty],$$

to the problem under consideration we get

$$\frac{\sin \lambda x}{\sin x} = \frac{\lambda\left[1-\left(\frac{\lambda x}{\pi}\right)^2\right]\left[1-\left(\frac{\lambda x}{2\pi}\right)^2\right]\left[1-\left(\frac{\lambda x}{3\pi}\right)^2\right]\cdots}{\left[1-\left(\frac{x}{\pi}\right)^2\right]\left[1-\left(\frac{x}{2\pi}\right)^2\right]\left[1-\left(\frac{x}{3\pi}\right)^2\right]\cdots}.$$

Consider now the ratio of the nth binomial factor of the numerator to the nth or corresponding factor of the denominator:

$$\frac{1-\left(\frac{\lambda x}{n\pi}\right)^2}{1-\left(\frac{x}{n\pi}\right)^2} = \frac{n^2\pi^2-\lambda^2x^2}{n^2\pi^2-x^2} = 1+\frac{(1-\lambda^2)x^2}{n^2\pi^2-x^2}.$$

For any chosen value of n (1, 2, 3, $\cdots$) the denominator of the last fraction decreases while the numerator increases as x grows larger. The fraction is always finite and positive because of the hypotheses $0 < \lambda < 1$ and $0 < x < \pi$. Consequently, the ratio of any binomial in the numerator of the expression for $(\sin \lambda x)/(\sin x)$ to the corresponding binomial in the denominator increases with x. It is accordingly manifest that the product of an infinite number of such converging ratios increases as x increases, and so the problem is solved.

II. Solution by Frank Irwin, University of California.

We shall show that, for the values of x in question, the derivative of the function is positive. This derivative is

$$\frac{\lambda\cdot\cos\lambda x\cdot\sin x-\sin\lambda x\cdot\cos x}{\sin^2 x}$$

or

$$\frac{1}{x\cdot\sin^2 x\cdot\sin\lambda x}(\lambda x\cdot\cot\lambda x - x\cdot\cot x).$$

This will be positive if $\lambda x\cdot\cot\lambda x > x\cdot\cot x$, that is, if $y\cdot\cot y$, let us say, continually decreases as y increases from 0 to π. This is so, since its derivative, $\cot y - y\csc^2 y$, or $(\sin 2y - 2y)/2\sin^2 y$, is always negative.

A solution similar to the second was received from Elijah Swift.

392. Proposed by HORACE OLSON, Student at The University of Chicago.

Two right cylinders of radii a and b, respectively, are placed so that their axes intersect at right angles. Find the volume common to them.

Solution by Elijah Swift, University of Vermont.

Assume $a > b$. Let the axis of the smaller cylinder be the z-axis, that of the larger, the y-axis. We see that the volume is

$$V = 8\int_0^b\int_0^{\sqrt{b^2-x^2}}\sqrt{a^2-x^2}\cdot dy\cdot dx = 8\int_0^b\sqrt{(a^2-x^2)(b^2-x^2)}\cdot dx.$$

This latter integral is elliptic, and may be expressed in terms of complete elliptic integrals of the first and second kinds. Letting $x = b\,\mathrm{sn}\,(y, b/a)$, the integral becomes

$$8ab^2\int_0^K\left\{1-\frac{a^2+b^2}{a^2}\mathrm{sn}^2 y+\frac{b^2}{a^2}\mathrm{sn}^4 y\right\}dy.$$

This, in turn, may be integrated by reduction formulas and gives finally for the volume,

$$V = \tfrac{8}{3}a\left[(b^2-a^2)K\left(\frac{b}{a}\right)+(a^2+b^2)E\left(\frac{\pi}{2},\frac{b}{a}\right)\right],$$

where K denotes the complete elliptic integral of the first kind, E the elliptic integral of the second kind.

Note.—For this same problem, see Byerly's *Integral Calculus*, Chap. XVI, Elliptic Integrals. In the 1902 edition, however, an incorrect answer is given.

Also solved by H. S. UHLER.

MECHANICS.

307. Proposed by LAENAS G. WELD, Pullman, Illinois.

Four forces W, X, Y, and Z, concurrent in O, are in equilibrium. Prove that

$$W : X : Y : Z :: \Delta_1 : \Delta_2 : \Delta_3 : \Delta_4,$$

where

$$\Delta_1 = \begin{vmatrix} 1 & \cos XOY & \cos XOZ \\ \cos XOY & 1 & \cos YOZ \\ \cos XOZ & \cos YOZ & 1 \end{vmatrix}^{\frac{1}{2}}; \quad \Delta_2 = \begin{vmatrix} 1 & \cos WOY & \cos WOZ \\ \cos WOY & 1 & \cos YOZ \\ \cos WOZ & \cos YOZ & 1 \end{vmatrix}^{\frac{1}{2}};$$

$$\Delta_3 = \begin{vmatrix} 1 & \cos WOX & \cos WOZ \\ \cos WOX & 1 & \cos XOZ \\ \cos WOZ & \cos XOZ & 1 \end{vmatrix}^{\frac{1}{2}}; \quad \Delta_4 = \begin{vmatrix} 1 & \cos WOX & \cos WOY \\ \cos WOX & 1 & \cos XOY \\ \cos WOY & \cos XOY & 1 \end{vmatrix}^{\frac{1}{2}}.$$

SOLUTION BY H. S. UHLER, Yale University.

For brevity let

$$l \equiv \cos WOX = \cos XOW, \quad p \equiv \cos XOY = \cos YOX,$$
$$m \equiv \cos WOY = \cos YOW, \quad q \equiv \cos XOZ = \cos ZOX,$$
$$n \equiv \cos WOZ = \cos ZOW, \quad r \equiv \cos YOZ = \cos ZOY.$$

Since the forces are in equilibrium the (algebraic) sum of their projections on any straight line must vanish. Hence, by projecting the forces successively upon the lines of action of W, X, Y, and Z, respectively, we obtain the following redundant set of homogeneous equations:

$$W + lX + mY + nZ = 0,$$
$$lW + X + pY + qZ = 0,$$
$$mW + pX + Y + rZ = 0,$$
$$nW + qX + rY + Z = 0.$$

By applying a well-known theorem of determinants to the cofactors of the first row we find

$$W : X :: \begin{vmatrix} 1 & p & q \\ p & 1 & r \\ q & r & 1 \end{vmatrix} : - \begin{vmatrix} l & p & q \\ m & 1 & r \\ n & r & 1 \end{vmatrix}, \quad \text{or} \quad W : X :: - \Delta_1^2 : \begin{vmatrix} l & p & q \\ m & 1 & r \\ n & r & 1 \end{vmatrix}. \tag{1}$$

Similarly, for the second line or row

$$W : X :: \begin{vmatrix} l & m & n \\ p & 1 & r \\ q & r & 1 \end{vmatrix} : - \begin{vmatrix} 1 & m & n \\ m & 1 & r \\ n & r & 1 \end{vmatrix}, \quad \text{or} \quad W : X :: \begin{vmatrix} l & m & n \\ p & 1 & r \\ q & r & 1 \end{vmatrix} : - \Delta_2^2. \tag{2}$$

The determinants constituting the fourth and third terms of proportions (1) and (2), respectively, are equal because the rows of one are the same as the corresponding columns of the other, hence the product of (1) and (2) is

$$W^2 : X^2 :: \Delta_1^2 : \Delta_2^2,$$

or, since we are only dealing with the arithmetical magnitudes of the forces,

$$W : X :: \Delta_1 : \Delta_2.$$

Obviously cyclical permutation of the symbols leads to the required proportions

$$W : X : Y : Z :: \Delta_1 : \Delta_2 : \Delta_3 : \Delta_4.$$

Remark: From the principles of statics we know that the four original scalar equations must be compatible or simultaneous and, from the theory of determinants, that the necessary and sufficient condition for this compatibility is the vanishing of the determinant Δ of the coefficients of W, X, Y, and Z. Hence, the six cosines l, m, n, p, q, r pertaining to a closed space quadrilateral are not all independent but are mutually connected by the relation $\Delta = 0$, that is,

$$l^2 + m^2 + n^2 + p^2 + q^2 + r^2 - l^2r^2 - m^2q^2 - n^2p^2 - 2lmp$$
$$- 2lnq - 2mnr - 2pqr + 2lmqr + 2lnpr + 2mnpq = 1.$$

310. Proposed by EMMA M. GIBSON, Drury College.

"A particle moveable on a smooth spherical surface of radius a is projected along the horizontal great circle with a velocity v which is great compared with $\sqrt{(2ga)}$. Prove that its path lies between this great circle and a parallel circle whose plane is at a depth $2ga^2/v^2$ below the centre, approximately."

From Lamb's *Dynamics*, p. 334, Ex. 3.

SOLUTION BY H. S. UHLER, Yale University.

Since the particle has only two degrees of freedom let us choose as coördinates the angles θ and ϕ which the radius terminating in the particle (of mass m) makes with any fixed vertical plane which contains the center of the sphere and with the horizontal plane passing through this center, respectively. Since none of the forces acting on the particle has a moment about the vertical diameter of the sphere it follows that the moment of momentum about this axis must remain invariable. At the instant of projection the moment of momentum equals $a \cdot mv$. At some later time the moment of momentum will be $r \cdot mr\dot{\theta}$ or $ma^2(\cos^2\phi)\dot{\theta}$, because $r = a\cos\phi$. Consequently

$$a(\cos^2\phi)\dot{\theta} = v. \tag{1}$$

Another relation between θ and ϕ may be derived from the principle of the conservation of energy. At the instant of projection the total energy equals $\frac{1}{2}mv^2 + V$, where V symbolizes the potential energy at the level of the center of the sphere. In general, at any later time the components of the kinetic energy in the horizontal and vertical planes will be respectively $\frac{1}{2}ma^2(\cos^2\phi)\dot{\theta}^2$ and $\frac{1}{2}ma^2\dot{\phi}^2$, while the potential energy will be $V - mga\sin\phi$. Therefore,

$$a^2(\cos^2\phi)\dot{\theta}^2 + a^2\dot{\phi}^2 - 2ag\sin\phi = v^2. \tag{2}$$

Substituting the value of $\dot{\theta}$ from (1) in (2) we find

$$a^2\dot{\phi}^2 = 2ag\sin\phi - v^2\tan^2\phi$$

or

$$a\dot{\phi} = (2ag\sin\phi - v^2\tan^2\phi)^{\frac{1}{2}}.$$

The last equation shows that ϕ cannot be negative with $\dot{\phi}$ real, in other words, the particle will not rise above the plane of projection. Writing this equation in the form

$$2a\sqrt{2ag}(\cos\phi)\dot{\phi} = [(\sin\phi)(\sqrt{v^4 + 16a^2g^2} + v^2 + 4ag\sin\phi)(\sqrt{v^4 + 16a^2g^2} - v^2 - 4ag\sin\phi)]^{\frac{1}{2}},$$

it becomes self-evident that the greatest value which ϕ can attain without making $\dot{\phi}$ complex is given by the formula

$$4ag\sin\phi_0 = \sqrt{v^4 + 16a^2g^2} - v^2.$$

The corresponding vertical distance below the horizontal diametral plane is $a\sin\phi_0 = d$. Then, rigorously,

$$d = \frac{1}{4g}(\sqrt{v^4 + 16a^2g^2} - v^2).$$

When $4ag/v^2$ is less than unity we may expand the radical by the binomial theorem to obtain an approximate rational expression for d. Then

$$d = \frac{v^2}{4g}\left[\left(1 + \frac{8a^2g^2}{v^4} - \frac{32a^4g^4}{v^8} + \cdots\right) - 1\right]$$

or, as a first approximation,

$$d = 2ga^2/v^2.$$

Remark: If $v = 0$, (1) gives $\dot{\theta} = 0$ or $\theta = constant$, showing that the motion is now uniplanar. Under these conditions (2) reduces to $a\dot{\phi}^2 = 2g \sin \phi$, since $\dot{\phi} \neq 0$. Then $a\ddot{\phi} = g \cos \phi$ or putting $\phi = (\pi/2) + \xi$, $a\ddot{\xi} = -g \sin \xi$ which is the familiar form of the equation of motion of a simple pendulum.

311. Proposed by B. J. BROWN, Student in Drury College.

A particle oscillates in a straight line about a center of force varying as the distance, and is subject to a retardation $k \times (\text{vel.})^2$. If a, b be two successive elongations, on opposite sides, prove that

$$(1 + 2ka)e^{-2ka} = (1 - 2kb)e^{2kb}.$$

What form does the result take if a is infinite?

From Lamb's *Dynamics*, p. 299, ex. 14.

I. Solution by H. S. Uhler, Yale University.

Let the displacement of the particle at any time t from the center of attraction be denoted by x. To fix the ideas, consider the forces acting on the particle of mass m at an instant when its displacement and velocity are both positive. Then

$$m\frac{d^2x}{dt^2} = -cmx - km\left(\frac{dx}{dt}\right)^2,$$

where c is a positive constant, or

$$\frac{d^2x}{dt^2} + k\left(\frac{dx}{dt}\right)^2 + cx = 0. \tag{1}$$

Equation (1) is a standard form of which the first integral can be obtained at once, for the solution of

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 f(y) + F(y) = 0$$

is

$$\frac{dy}{dx}e^{\int f(y)dy} = \left\{A - 2\int dyF(y)e^{2\int f(y)dy}\right\}^{\frac{1}{2}}.$$

See Forsyth's *Differential Equations*, third ed., Art. 67.

Hence

$$\frac{dx}{dt}e^{kx} = \left\{A + \frac{ce^{2kx}}{2k^2}(1 - 2kx)\right\}^{\frac{1}{2}}, \tag{2}$$

The constant of integration A in equation (2) may be found from the datum that $dx/dt = 0$ when $x = -a$, whence

$$\frac{dx}{dt}e^{kx} = \frac{1}{k}\left\{\frac{c}{2}\left[(1 - 2kx)e^{2kx} - (1 + 2ka)e^{-2ka}\right]\right\}^{\frac{1}{2}}. \tag{3}$$

Again, by hypothesis, $dx/dt = 0$ when $x = b$ so that equation (3) gives the required relation

$$(1 + 2ka)e^{-2ka} = (1 - 2kb)e^{2kb}.$$

When a becomes infinite we have the so-called indeterminate form

$$\frac{1 + 2ka}{e^{2ka}} = 0 + \frac{z}{e^z} = \frac{\infty}{\infty}.$$

In general,

$$\frac{z}{e^z} = \left[z^{-1} + 1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots + \frac{z^n}{(n+1)!} + \cdots\right]^{-1};$$

consequently as z increases indefinitely ze^{-z} approaches zero as a limit. The same result follows from the well-known fact that "any infinite number is an infinity of higher order than any power of its logarithm."

Finally, since b is not negative, $e^{2kb} \neq 0$ and

$$b = \frac{1}{2k}.$$

Remark: If k were zero the last result would become $b = \infty$ which is consistent with the fact that equation (1) now represents rectilinear simple harmonic motion with infinite amplitude $a = b = \infty$.

II. Solution by Horace Olson, Chicago, Illinois.

From the conditions of the problem, $dv/dt = -cx \mp kv^2$, if the center of force is at $x = 0$. The upper or lower sign of kv^2 is taken according as the particle is moving in a positive or a negative direction.

From the identity $dv/dt = dv/dx \cdot dx/dt = vdv/dx$, we have $vdv/dx = -cx \mp kv^2$. Set $w = v^2$. Then $dw/dx = -2cx \mp 2kw$, a linear differential equation. Whence,

$$we^{\pm 2kx} = \mp \frac{cxe^{\pm 2kx}}{k} + \frac{ce^{\pm 2kx}}{2k^2} + C.$$

v, and, therefore, $w = 0$ when $x = a$ or b. Hence,

$$0 = \left(\frac{ac}{k} + \frac{c}{2k^2}\right)e^{-2ka} + C = \left(-\frac{bc}{k} + \frac{c}{2k^2}\right)e^{2kb} + C;$$

whence

$$(1 + 2ka)e^{-2ka} = (1 - 2kb)e^{2kb}.$$

When $a = +\infty$, the first member of this equation becomes 0 and $(1 - 2kb)e^{2kb} = 0$. Hence, if b is not $-\infty$, $2kb = 1$.

When $a = -\infty$, we have $-\infty = (1 - 2kb)e^{2kb}$; whence $b = +\infty$.

QUESTIONS AND DISCUSSIONS.

Send all communications to U. G. Mitchell, University of Kansas, Lawrence, Kansas.

NEW QUESTIONS.

32. In a discussion of the Peaucellier[1] Cell by analytic methods the following equations are obtained:

$$(1)\ (x_2 - x_1)^2 + (y_2 - y_1)^2 - b^2 = 0; \qquad (2)\ (x_3 - x_1)^2 + (y_3 - y_1)^2 - b^2 = 0;$$

$$(3)\ (x_2 - X)^2 + (y_2 - Y)^2 - b^2 = 0; \qquad (4)\ (x_3 - X)^2 + (y_3 - Y)^2 - b^2 = 0;$$

$$(5)\ x_2^2 + y_2^2 - K^2 = 0; \qquad (6)\ x_3^2 + y_3^2 - K^2 = 0;$$

$$(7)\ x_1^2 + y_1^2 - 2cx_1 = 0.$$

The result of eliminating $x_1, y_1, x_2, y_2, x_3, y_3$ gives an equation of the first degree, which establishes that the linkage will trace a straight line. There are various ways of effecting this elimination.

I. What element of the situation is left unused by the following procedure in the elimination?

(a) From equations (1), (3), (5) eliminate x_2 and y_2 and obtain an equation

$$(8)\qquad f_1(x_1, y_1) = 0.$$

(b) From equations (2), (4), (6) eliminate x_3 and y_3, and obtain an equation

$$(9)\qquad f_2(x_1, y_1) = 0.$$

(c) From equations (7), (8), (9) eliminate x_1 and y_1, and obtain the desired equation.

II. How should this procedure be supplemented to secure the result?

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[1] If reference is made to the article on "Linkages" in the December (1915) Monthly by Mr. Leavens, the following coördinates may be applied to his figure: $O(o, o)$; $C(c, o)$; $P_1(x_1, y_1)$; $M(x_2, y_2)$; $M_1(x_3, y_3)$; $P_2(XY)$.

DISCUSSIONS.

RELATING TO INDETERMINATE FORMS.

I. A QUESTION WITH RESPECT TO 0/0.

By J. W. Nicholson, Louisiana State University.

Has 0/0, (*a*) in particular cases a definite value, or (*b*) is it always indeterminate?

A generation ago mathematicians, in general, answered (*a*) affirmatively and (*b*) negatively; but in recent years, as a rule, they do just the reverse. This change is due to the ruling out of division by zero, because, as is asserted, it is impossible in cases like $a/0$ and indeterminate in cases like $0/0$.

Without advocating the one contention above the other, I invite a careful consideration of the following argument, hoping that some one will point out the fallacy, if fallacy there be.

The locus of

$$y = \frac{x^2 - a^2}{x - a} \tag{1}$$

consists of the two straight lines[1]

$$y - x - a = 0 \tag{2}$$

and

$$x - a = 0, \tag{3}$$

as shown by clearing of fractions, transposing and factoring.

Now when $x = a$ in (1), *y has a definite and also an indeterminate value.*

For, according to the definitions of loci and their equations, the value of y when $x = a$ is the ordinate of the point of intersection of the loci of $x - a = 0$ and (1); that is, of the line $x - a = 0 \cdots$ (4) and the two lines (2) and (3).

But the point of intersection of (4) and (2) is $(a, 2a)$, and that of (4) and (3) is $(a, 0/0)$.

Therefore, when $x = a$ in (1) we have

$$y = 2a \quad \text{and} \quad y = 0/0.$$

It will be seen that the above process of determining the definite value $2a$ is the simple operation of finding the intersection of two given lines, and is independent of the idea or doctrine of limits.

By the same method the following more general proposition may be proved:

When $x = x'$ the definite value of

$$\frac{f(x) - f(x')}{x - x'} \quad \text{is} \quad f_1(x),$$

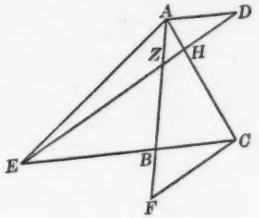
in which $f_1(x)$ is the derivative of $f(x)$.

[1] See, for instance, J. W. Young's *Fundamental Concepts of Algebra and Geometry*, p. 214; Granville's *Differential and Integral Calculus*, p. 171; Townsend and Goodenough's *Essentials of Calculus*, p. 21; McMahon and Snyder's *Differential Calculus*, p. 116; and many others.

II. A GEOMETRICAL ILLUSTRATION OF THE FORM ∞/∞.

By James H. Weaver, High School, West Chester, Pa.

In the American Mathematical Monthly, Vol. XXIII (1916), pp. 41–44, Professor Lovitt has given some interesting illustrations of indeterminate forms. The following is an example of the form ∞/∞ and involves the invariance of cross-ratio. The example and the proof have been taken from Pappus.[1] Some



slight modifications in the statement and proof of the theorem have been made to suit the present needs.

Theorem. *Let there be four lines, AE, AB, AC and AD concurrent in A, and let these be cut by a line EZHD and let EBC be drawn parallel to AD. Then* $ED \cdot ZH : EZ \cdot DH = CB : BE$.

Proof. Through C draw CF parallel to ED, cutting AB produced in F. Since CF is parallel to HZ, $CA : AH = CF : ZH$, and since EC is parallel to AD, $CA : AH = ED : DH$. Therefore $ED : DH = CF : ZH$ and $ED \cdot ZH = CF \cdot DH$. Hence

$$\begin{aligned} ED \cdot ZH : EZ \cdot DH &= CF \cdot DH : EZ \cdot DH, \\ &= CF : EZ, \\ &= CB : BE. \end{aligned}$$

Now AD and BC will intersect at infinity and because of the invariance of the cross-ratio under a projective transformation, we have

$$ED \cdot ZH : EZ \cdot DH = CB \cdot \infty : BE \cdot \infty.$$

In this connection it is interesting to note that the above theorem is the foundation for the celebrated Configuration of Pappus.[2]

RELATING TO THE MECHANICAL TRISECTION OF AN ANGLE.

By Bernhart Ingimundson, Student in the University of Manitoba.

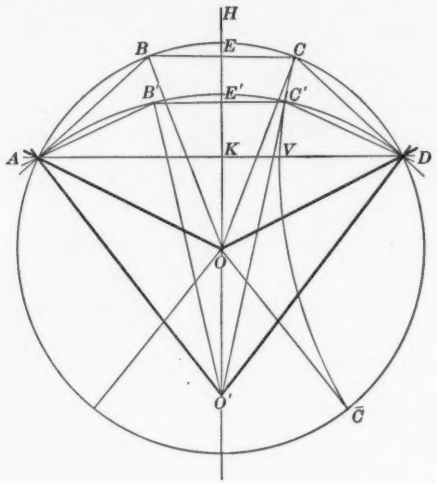
Note.—Interest in the trisection problem is perennial. It was old in the time of Plato and is new today. It is probably worth while to republish solutions occasionally in order to remind readers that the problem is readily solvable by means other than ruler and compass. Trisection

[1] Pappus Alexandrinus, *Collectio*, ed. Hultsch, p. 882.

[2] Cf. Veblen and Young, *Projective Geometry*, Vol. I, p. 99.

by means of the hyperbola dates back to the time of Pappus (about 300 A.D.) and the use of the hyperbola of eccentricity 2 is by no means new;[1] but we believe many of our readers will enjoy the solution given below, sent in by a first year engineering student.—EDITOR.

Let AOD be any angle, acute, obtuse or reflex. With radius OA describe a circle intersecting the sides of the angle at A and D. Suppose the arc AD trisected at points B and C and join AB, BC and CD. Bisect the angle AOD by the line OH, cutting BC at E. Then OH is the perpendicular bisector of BC and AD. Hence $CE = CD/2$.



Similarly, if O' be any other point on OH and the angle $AO'D$ be trisected by $O'B'$ and $O'C'$, $C'E' = C'D/2$. Hence, as the point O moves along the line OH the point C describes a curve such that the ratio of CE to CD is constant and equal to $\frac{1}{2}$. Therefore the locus of C is an hyperbola with focus at D, directrix OH and eccentricity 2. Similarly, it may be shown that the reflex angle AOD is trisected by $O\bar{C}$.

We have, therefore, a mechanical means of trisecting any given angle AOD as follows:

Cut off $OA = OD$ and draw OH perpendicular to and bisecting AD at K.

With D as focus and OKH as directrix construct an hyperbola of eccentricity 2 cutting the circle whose center is O and radius OD at C.

CO then trisects the given angle AOD.

The required hyperbola could be constructed by the ordinary pin, thread and ruler method, or, if one uses the same length of chord AD in each case, then the arc of an hyperbola which is such that the distance from the focus to the vertex is $\frac{1}{3}AD$ may be used to obtain the required division.

[1] Cf. articles by Candy in *Kansas University Quarterly*, Vol. II (1893), pp. 35–45; Crawford, *Edinburgh Math. Soc. Proc.*, Vol. XVI, pp. 42–45; Genese, *Messenger of Math.*, Vol. I (1872), pp. 103, 181; and various others, which may be readily found.

SUMMER MEETING OF THE ASSOCIATION.

In response to the cordial invitation of the Massachusetts Institute of Technology, the Council has decided to announce a summer meeting of the Association in Cambridge, Massachusetts. The meeting will be held Friday and Saturday, September 1 and 2, in the new buildings of the Massachusetts Institute of Technology. These dates immediately precede the dates for the summer meeting of the American Mathematical Society. It is hoped that the attendance at both meetings may thus be large. Specially low round-trip rates are already announced from Chicago to Boston, good till October first, and doubtless equally favorable rates will be in effect from other western points.

Plans are in preparation for social meetings on Friday and Saturday evenings, and perhaps for a nearby, week-end excursion between the meetings. Arrangements will be made for luncheon on Friday and Saturday, and information in regard to hotels and boarding places in Boston will be furnished on application to Professor H. W. TYLER, Chairman of the committee on arrangements, Massachusetts Institute of Technology.

The program committee consists of Professor H. B. FINE of Princeton University, Professor C. S. SLICHTER of the University of Wisconsin, and Professor DUNHAM JACKSON of Harvard University. A preliminary announcement of the program will be made at an early date, and a fuller announcement may be ready in time for printing in the June issue.

THE LIBRARY OF THE ASSOCIATION.

As announced earlier, it is the purpose of the Association to organize a library as rapidly as possible. Plans are already under way for placing the facilities of such a library within the reach of any member of the Association who may desire to take advantage of it. Further announcement of particulars along this line will be made at an early date. Meanwhile, any authors of books or publishers who are willing to contribute volumes to the Association Library may send them directly to Professor W. D. CAIRNS, Oberlin College, Oberlin, Ohio. The following books have already been presented for this purpose: Robert of Chester's Latin Translation of the Algebra of Al-Khowarizmi, by LOUIS C. KARPINSKI, University of Michigan, published by the Macmillan Company; the Napier Tercentenary Memorial Volume, edited by CARGILL G. KNOTT, published for the Royal Society of Edinburgh by Longmans, Green and Company; Fundamental Conceptions of Mathematics, by ROBERT P. RICHARDSON and EDWARD H. LANDIS, published by the Open Court Publishing Company; and Fundamental Sources of Efficiency, by FLETCHER DURELL, published by J. B. Lippincott Company.

As previously stated, a complete set of the MONTHLY has been made available through the courtesy of Professor E. H. MOORE, who has been a subscriber since the establishment of the journal in 1894. These volumes will be bound and placed in the Association Library at an early date.

Unfortunately, complete sets of Volume XX (the first year after the reorganization of the MONTHLY) are nearly exhausted, so that it will be impossible to supply any more libraries or individuals with these volumes, though the calls for them are numerous. But it is proposed to bind the few sets of this volume which still remain and place them in the library in order that they may be loaned to members who may desire to see them. Sets of Volumes XXI and XXII are also running low, but several more may be had at two dollars and fifty cents per volume by those who apply early for them. It has been our purpose to make the issues of Volume XXIII sufficiently large to cover all contingencies; but even here, the unexpected demands have drawn heavily on our supply.

No complete sets of the MONTHLY for the years preceding the reorganization in 1913 are available, but numerous separate volumes and single copies can be had at two dollars per volume or twenty-five cents for single copies; and the Managing Editor will be pleased to assist members of the Association, as far as possible, in completing broken sets or volumes.

NOTES AND NEWS.

SEND ALL COMMUNICATIONS TO D. A. ROTHROCK, Indiana University, Bloomington, Ind.

Dr. R. W. BURGESS, of Cornell University, has been appointed instructor in mathematics at Brown University.

In *Science* for April 21 is a four-page paper on "A new method for the graphic solution of algebraic equations," by Professor H. G. DEMING, of the University of the Philippines.

At the spring meeting of the New England Association of Mathematical Teachers held at the Massachusetts Institute of Technology on May 6, Professor R. C. Archibald read a paper on "Euclid and his works."

The one hundred eighty-fourth regular meeting of the American Mathematical Society was held in New York City on Saturday, April 29, 1916. Twenty-six papers were presented at the two sessions. There were present fifty-one members.

Doctor HENRY M. SHEFFER is instructor in philosophy at the College of the City of New York. Doctor Sheffer is a member of the American Mathematical Society and of the Mathematical Association of America and has many interests on the mathematical side of philosophy.

The Benjamin Peirce instructorships at Harvard University have been filled by the appointments of Dr. E. KIRCHER and Dr. W. L. HART. The latter has just received the doctorate at the University of Chicago.

At Sheffield Scientific School of Yale University, Dr. E. J. MILES has been promoted to an assistant professorship of mathematics. Dr. Miles graduated at Indiana University in 1906, was granted the A.M. degree at Swarthmore College in 1907, and received the doctorate at the University of Chicago in 1910.

On the evening of March 4, 1916, Dean GEORGE D. OLDS, head of the department of mathematics in Amherst College, spoke at Mount Holyoke College to the students in the mathematics classes. His subject was "Reminiscences of my German teachers in the early eighties."

The sixth regular meeting of the American Mathematical Society at Chicago, being the thirty-seventh regular meeting of the Chicago Section, was held at the University of Chicago, Friday and Saturday, April 20 and 21, 1916. There were present forty-six members. Thirty papers were presented.

At the March meeting of the Brown University Mathematical Club, sixty-eight members were present, and Professor N. F. DAVIS presided. The following papers were presented by the students: "History and mathematics of the sundial," by Miss RUTH H. HALL; "Linkages with numerous models," by B. H. BROWN.

"The Mathematical Association of America," is the title of an article by President E. R. HEDRICK in *School and Society* for March 11, 1916. In this paper President Hedrick sets forth the history culminating in the organization of the Association, especially in its relation to the American Mathematical Society.

Miss OLIVE C. HAZLETT has published her Doctor's dissertation in the *American Journal of Mathematics*. The title is: "On the classification and invariantive characterization of nilpotent algebras." Doctor Hazlett is to begin work as a member of the faculty of Bryn Mawr College in September, 1916.

An indication of the great decrease in the number of new publications in the natural and exact sciences, occasioned by the European war, is the size of Vol. 37 (1915) of "Naturæ Novitates." It contains only 340 pages, as compared to more than 500 pages in 1914, and upwards of 620 pages each for the years of 1912 and 1913.

Upon the occasion of the celebration of the seventieth birthday of the distinguished Swedish mathematician, Professor M. G. MITTAG-LEFFLER, he and his wife set aside their entire fortune to be used in founding an international institute for pure mathematics. The *Acta Mathematica* has been edited by Professor Mittag-Leffler since its founding in 1882.

The first meeting of the Ohio Section of the ASSOCIATION was held at Columbus, on April 21, 22, 1916. This section was formed at Columbus at the same time that the national association was organized and application for admission was made then. A full report of this meeting will be printed in the June issue.

"Studies on Divergent Series and Summability," is the title of a monograph by Professor W. B. FORD, which constitutes Volume II of the Scientific Series of Publications by the University of Michigan. These studies are published by the Macmillan Company. The first volume, which was recently reviewed in the MONTHLY was "Robert of Chester's Latin translation of the algebra of Al-Khowarizmi," by Professor KARPINSKI.

In the publications of the General Education Board, ABRAHAM FLEXNER has discussed the topic, "A Modern School," in which his reference to mathematics is much along the same line as that of some other writers, such as the commissioner of education of Massachusetts and the state superintendent of New Hampshire. This document is quite fully outlined in *School and Society* for April 22, 1916.

A number of reprints have been received from the *Proceedings of the Edinburgh Mathematical Society*, entitled: "Research papers of the mathematical department of the University of Edinburgh." The authors are: E. T. WHITTAKER, E. LINDSAY RICE, L. R. FORD, A. G. BURGESS, EDWARD BLADES, and ARCHIBALD MILNE. This mathematical laboratory is under the direction of Professor E. T. WHITTAKER, and the publications relate for the most part to the field of applied mathematics.

The initial number of a mathematical periodical, *Revista de Matematicas*, edited by MANUEL GUITARTE, Buenos Aires, Argentina, appeared recently, and contains articles as follows: "Mathematics, its nature and importance," by C. C. DASSEN; "On algebraic division," by E. REBUELTO; "Questions of elementary mathematics related to the theory of groups and the differential calculus," by J. DUCLOUT; "The potential function in hyperspace," and "Area of rectilinear triangles, some particular cases," by M. GUITARTE. The *Revista* is published in the Spanish language and is to appear bi-monthly.

At the Kansas State Agricultural College, a mathematical club has been organized and in operation for several years. The club is composed of students and members of the departmental faculty; it meets bi-weekly and presents interesting programs upon pedagogical, historical and applied subjects. The printed program for the current year shows a very interesting selection of topics. The membership as indicated upon the program consists of one hundred and twenty-four persons.

At the twenty-eighth Educational Conference of the academies and high schools in relations with the University of Chicago, on Friday and Saturday, April 14 and 15, 1916, the mathematical section, which was in charge of Professor J. W. A. YOUNG, had for its general subject, "The consideration of the relative importance of topics usually treated in algebra and plane and solid geometry with reference especially to the credit value of the unit." There were also

discussions of current committee reports on the "definition of the units in mathematics" and on "mathematics in the secondary school of tomorrow."

The Mathematical Club of Smith College was organized in 1899. Its membership is limited to students of the senior and junior classes who are carrying major courses in mathematics. Professor Cushing is president and other members of the faculty and the graduates living in the city are admitted to honorary membership. The club meets once in three weeks for the consideration of papers presented by the students or the faculty, and two social meetings are held annually. During the last year a study was made of the history of mathematics, considering first the general development of the subject, and later some of the great names in mathematical literature.

Mr. PHILIP E. B. JOURDAIN, of Fleet, near London, contributes to *Mind* (Vol. 25, pp. 42–55) an article on "Zeno's Flying Arrow: An Anachronism," in which Zeno is represented in hades as discussing with Socrates the views regarding motion held in ancient and modern times. Zeno comments even on Bertrand Russell's Lowell Lectures of 1914. The article is extremely entertaining. Jourdain holds that Zeno desired to combat the Pythagorean belief that lines are made up of points. The version of the Arrow by Sextus Empiricus, "a man never dies, for if a man die, it must be either at a time when he is alive or at a time when he is not alive," is compared with the claim of the Chinese philosopher, Hui Tzŭ, that "a motherless colt never had a mother; for when it had a mother it was not motherless, and at every other moment of its life it had no mother."

At the meeting of the Iowa Academy of Science in Des Moines on April 27, the members of the Mathematical Association of America who were present held a conference and decided to petition the Council for the establishment of an Iowa section. An organization was effected with Professor A. G. SMITH, of the State University, as president, Professor G. A. CHANEY, of the State College of Agriculture and Mechanic Arts, as vice-president, and Professor I. F. NEFF, of Drake University, as secretary-treasurer. Professor R. B. McClenon of Grinnell College, and Professor J. F. Riley of the State University were appointed a committee on membership. The officers were instructed to prepare and present a petition to the Council for the establishment of the Iowa Section, the petition to be sent for signatures to all the thirty-two members of the ASSOCIATION resident in the State of Iowa.

"Constructive Geometry" is the title of a pamphlet of 75 pages by E. R. HEDRICK, recently published by the Macmillan Company. It is in the form of a notebook with blank pages interspersed. The constructions and suggestions are exceedingly interesting and well organized. The following quotation from the preface adequately states the character and purpose of the work: "The saying is trite that students who enter formal courses in Euclidean Geometry have to learn both the strange methods of formal logic and the equally strange geometric forms. A course to acquaint students with the elementary forms and

constructions is valuable particularly to those who never go on to a more formal course and it furnishes a basis for a truer comprehension by those who do go on. Such courses are deservedly popular in Europe, but no good American geometric notebook exists. This is modeled after those long used successfully in England, some of which have been extensively used in America."

The Twenty-First Summer Meeting and Eighth Colloquium of the American Mathematical Society will be held at Harvard University during the week beginning Monday, September 4, 1916. The first two days will be devoted to the regular sessions for the presentation of papers. The Colloquium will open on Wednesday morning and close on Saturday morning. Two courses of five lectures each will be given by Professors G. C. EVANS, of Rice Institute, and OSWALD VEBLEN, of Princeton University. A brief outline of the courses is given in the May issue of the *Bulletin*. The year 1916 marks the twenty-fifth anniversary of the broadening out of the Society toward a national organization and of the founding of the *Bulletin*. It is proposed to arrange an appropriate celebration of this event at the summer meeting. Some seventy-five of the members of the Society in the year 1891 have retained their membership through these twenty-five years. Many of these will, doubtless, attend the celebration, which is itself a notable milestone in the Society's progress. The younger generation, on whom rests the Society's future, should also be there in large numbers, to take over the responsibility of guiding the destinies of the Society for the next quarter century.

The annual meeting at Ann Arbor of the Mathematics Section of the Michigan Schoolmasters' Club was opened by a luncheon Thursday noon, March 30, Newberry Hall, at which some sixty were present. Several five-minute talks contributed to the pleasure of the occasion. After the luncheon the meeting adjourned to the auditorium of the New Science Hall, where Professor Karpinski presented an address, illustrated by stereopticon, on The Story of Algebra. A second meeting was held on Friday at which several short papers were presented on the relation of higher mathematics to elementary mathematics. Professors Glover, Bradshaw, Running, Field, and Dr. Hopkins participated in the discussion, with reference to actuarial mathematics, projective geometry, calculus, mechanics and analytical geometry, respectively. The remainder of the program was devoted to various phases of the teaching of high-school mathematics. Of particular interest was an account of the continuation-school work at the Cass Technical School, with pupils drawn largely from the workers in the Detroit automobile plants. Mr. Lewis Hayes, an experienced and graduate engineer, who is in charge of the work in mathematics, gave an illuminating account of the preparation in arithmetic, algebra, geometry, and trigonometry which is actually required by the skilled workers in manufacturing plants. At the meeting next year it is proposed to discuss the question of forming a Michigan Section of the Mathematical Association of America. Mr. John Craig, of Muskegon, and Mr. W. V. Garretson, of Ann Arbor, were elected chairman and secretary, respectively, of the Section for the coming year.

IMPORTANT NOTICE TO FORMER MONTHLY SUBSCRIBERS.

Henceforth, the subscription price of the MONTHLY will be *three dollars net to all non-members of the* ASSOCIATION.

The following adjustments are proposed for those subscribers who failed to apply for membership in the ASSOCIATION before April first.

(1) Any person who had already paid $2.00 on the old basis for the year 1916 is asked to send *one dollar additional* to complete the subscription. If such a person desires also to become a member of the ASSOCIATION, an initation fee of $2.00 must now be paid.

(2) Those whose subscriptions for 1916 are entirely unpaid will now pay $3.00 for the subscription or $5.00 for membership in the ASSOCIATION, including the subscription.

In the case of subscriptions under (1) or (2) which expire *before* the end of 1916, please add *thirty cents extra* for each copy needed to complete the year. *Hereafter all subscriptions will date from January of each year.*

(3) An institution in which the Calculus is taught may become an *institutional member* of the ASSOCIATION by the payment of *five dollars* annually, which will entitle the library to receive *two copies of the* MONTHLY and the institution to send a voting delegate to all meetings of the ASSOCIATION. Institutions in which the Calculus is taught, whose libraries have already renewed their subscriptions for 1916, are asked to send *three dollars additional* and thus become institutional members of the ASSOCIATION.

Other institutions, and those not wishing to become institutional members, whose library subscriptions have already been renewed for 1916, are asked to send *one dollar additional* to complete the advanced price of the MONTHLY. No further subscriptions will be received at the old rate of two dollars, *and no discount from the advanced rate of three dollars will be allowed on subscriptions under any circumstances.*

(4) The obligations of the MONTHLY for 1916 will, of course, be fulfilled on the former basis in the case of any individual or institution whose subscription has already been paid, and who may decline to make the adjustment on the new basis, but it is hoped that all will comply, since even the advanced rate is only the actual cost of producing the journal.

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